Notes Lecture 7 (Sheaf Cohomology)

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We assume all categories to be small. Let $\mathcal{C}$ be a category. We denote the objects of $\mathcal{C}$ as $\mathcal{C}_0$. We let $*$ denote the terminal object of the category that we work in.

Goals:

- Small recap of the last 2 lectures
- Exactness of a sequence of sheaves and exactness on stalks
- Generalization of sheaf cohomology

1 Recap

The intend of this chapter is to do a short recap of the 'ingredients' we needed to define sheaf cohomology in the previous lecture.

Definition 1. (Global sections functor) Let $\mathcal{E}$ be a topos, then we let $\Gamma(-)\colon \mathcal{E} \to \mathrm{Set}$ denote the map $\Gamma(-) = \mathrm{Hom}_{\mathcal{E}}(*, -)$. We call this map the *global sections functor.*

As mentioned in Lecture 4, the global sections functor is right adjoint to the functor Const: Set $\to \mathcal{E}$, which is the canonical Set-tensoring functor applied to the terminal object, i.e. $S \mapsto \coprod_{s\in S} *$. If $\mathcal{E} = \mathrm{Sh}(\mathcal{C}, J)$ and S a set, then $\mathrm{Const}(S)$ is precisely the constant sheaf on S. Since Const preserves finite limits, we have that

$$\mathcal{E} \underset{\Gamma}{\overset{\mathrm{Const}}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathrm{Set}$$

is a geometric morphism.

For a category $\mathcal{C}$, we can look at the abelian group objects $\mathrm{Ab}(\mathcal{C})$ of $\mathcal{C}$, i.e. objects of $\mathcal{C}$ together with a multiplication, neutral element and inverse morphism, that respect commutativity. The following proposition gives us a characterization of the abelian group objects of a (pre)sheaf category .

Proposition 2. Let $\mathcal{C}$ be a small category with finite products, and let $(\mathcal{C}, J)$ be a site. Then $\mathrm{Ab}(\mathrm{PSh}(\mathcal{C})) \simeq \mathrm{PSh}(\mathcal{C}; \mathrm{Ab})$ and $\mathrm{Ab}(\mathrm{Sh}(\mathcal{C}, J)) \simeq \mathrm{Sh}(\mathcal{C}, J; \mathrm{Ab})$, i.e. the abelian group objects of a category of (pre)sheaves are canonically *isomorphic* to the (pre)sheaves that map to abelian groups.

Proof. For $\mathrm{PSh}(\mathcal{C})$, it is not hard to check that this is the case, but it is quite tedious. For sheaves, there is even more trouble to go through. Because of this, the proof is left as an exercise to the reader. :) □

Moreover, recall that a geometric morphism $F\colon \mathcal{E} \to \mathcal{F}$ induces an adjunction

$$\mathrm{Ab}(\mathcal{E}) \underset{\mathrm{Ab}(F_*)}{\overset{\mathrm{Ab}(F^*)}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathrm{Ab}(\mathcal{F})$$

In general, $\mathrm{Ab}(F^*)$ need not preserve finite limits. However, for the global sections functor Γ and the map Const, we actually have that Ab(Const) preserves finite limits, so it is exact. We will use $\mathrm{Ab}(\Gamma)$ to define sheaf cohomology.

Lastly, to define sheaf cohomology, we needed right derived functors. For that, we need to define cohomology for objects of abelian categories.

Definition 3. Let $\mathcal{A}$ be an abelian category, and let $C^\bullet \in \mathrm{Ch}(\mathcal{A})$ be a cochain complex, i.e. a collection of objects C^i and maps ∂^i such that we get a long exact sequence

$$\cdots \longrightarrow C^{i-1} \xrightarrow{\partial^{i-1}} C^i \xrightarrow{\partial^i} C^{i+1} \longrightarrow \dots .$$

Then we define the *i-th (chain) cohomology* of $C^\bullet$ as $H^i(C^\bullet) \coloneqq \ker \partial^i / \mathrm{im} \partial^{i-1}$, for $i \in \mathbb{Z}_{\geq 0}$ (here, $C^{-1} = 0$ and $\partial^{-1} = 0$).

Let $\mathcal{A}$ be an abelian category that has enough injectives. Then for any $A \in \mathcal{A}_0$, there exists an injective resolution $0 \to A \to I^0 \to I^1 \to \dots$, i.e. a long exact sequence such that I^i is injective, that is $\mathrm{Hom}_{\mathcal{A}}(-, I^i)$ is exact, for all $i \in \mathbb{Z}_{\geq 0}$. We denote this resolution by $I^\bullet$.

Definition 4. (Right derived functor). Let $\mathcal{A}, \mathcal{B}$ be abelian categories, let $F \colon \mathcal{A} \to \mathcal{B}$ be a left exact functor and suppose $\mathcal{A}$ has enough injectives. Let $A \in \mathcal{A}_0$ and let $I^\bullet$ be an injective resolution of A. Then we define the *right derived functor* for $i \in \mathbb{Z}_{\geq 0}$ by $R^i F(A) \coloneqq H^i(F I^\bullet)$.

Note that, by definition, $R^0 F(A) = FA$. Moreover, by Homework 6, it holds that the above definition is well-defined.

Recall that, for a topos $\mathcal{E}$, $\mathrm{Ab}(\mathcal{E})$ has enough injectives. Now, we can define sheaf cohomology, as was done towards the end of last lecture.

Definition 5. (Sheaf Cohomology). Given a topos $\mathcal{E}$ and an object $F \in \mathrm{Ab}(\mathcal{E})_0$, we define the *i-th sheaf cohomology* of F to be the i-th right derived functor of the global sections functor $\mathrm{Ab}(\Gamma) \colon \mathrm{Ab}(\mathcal{E}) \to \mathrm{Ab}$, i.e. $H^i(F) \coloneqq H^i(\mathcal{E}, F) \coloneqq R^i \mathrm{Ab}(\Gamma) F$.

For intuition, the idea is that sheaf cohomology measures the 'failure of Γ to be exact'. In the next section, we will look more closely at what it means for a sequence of sheaves on a site to be exact.

2 Exactness of a sequence

Let $(\mathcal{C}, J)$ be a site and $\mathcal{E} = \mathrm{Sh}(\mathcal{C}, J)$. We want to analyse what it means for a sequence

$$0 \to A \to B \to C \to 0$$

in $\mathrm{Ab}(\mathcal{E}) \simeq \mathrm{Sh}(\mathcal{C}, J; \mathrm{Ab})$ to be exact. We will find out that left exactness can be computed 'object-wise'. However, for right exactness, we need to be a bit more specific.

Proposition 6. Let $\mathcal{E} = \mathrm{Sh}(\mathcal{C}, J)$ as above and let $A, B, C \in \mathrm{Ab}(\mathcal{E})_0$. Then the following are equivalent:

1. The sequence $0 \to A \to B \to C$ is exact.
2. For any $U \in \mathcal{C}_0$, the sequence $0 \to A(U) \to B(U) \to C(U)$ is exact.

Proof. Let $(-)^\# \colon \mathrm{PSh}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{C}, J)$ denote the sheafification functor. Then recall from Lecture 2 that we have the following geometric morphism:

$$\mathrm{Sh}(\mathcal{C}, J) \underset{i}{\overset{(-)^\#}{\underset{\longrightarrow}{\overset{\longleftarrow}{\perp}}}} \mathrm{PSh}(\mathcal{C})$$

This induces a geometric morphism on the abelian valued sheaves. Thus, since both i and $(-)^{\#}$ are left exact, we have that the sequence in 1. is exact precisely when it is exact as a sequence of presheaves. We then apply the fact that limits (and also colimits) of presheaves can be computed object-wise, yielding the equivalence between the given statements. □

Sadly, we cannot generalize Proposition 6 for right exact sequences of sheaves, since the sheafification functor is not necessarily right exact. Thus, we would want to have a characterization for epimorphisms which are preserved by the sheafification functor. These maps are precisely the local epimorphisms.

Definition 7. Let $\mathcal{F}, \mathcal{G} \in \mathrm{PSh}(\mathcal{C})$ be presheaves and let $f\colon \mathcal{F} \to \mathcal{G}$ be a morphism. Then f is called a *local epimorphism* if for every $U \in \mathcal{C}_0$ and every $s \in \mathcal{G}(U)$, there exists a covering family $(U_i \to U)_{i \in I}$ of U such that there exists an $s_i \in \mathcal{F}(U_i)$ such that $f(s_i) = s$ for any $i \in I$.

Proposition 8. Let $\mathcal{E} = \mathrm{Sh}(\mathcal{C}, J)$ be a topos, let $A, B \in \mathcal{E}_0$ and let $f\colon A \to B$ be a morphism. Then f is an epimorphism of sheaves if and only if f is a local epimorphism as a morphism of presheaves.

Proof. See Corollary 2.1.6 of [1] □

The above definition and proposition also hold for abelian valued (pre)sheaves. By combining the previous two propositions, we get the following:

Proposition 9. Let $\mathcal{E} = \mathrm{Sh}(\mathcal{C}, J)$ be a topos and let $A, B, C \in \mathrm{Ab}(\mathcal{E})_0$. Then the following are equivalent:

1. The sequence $0 \to A \to B \to C \to 0$ is exact.
2. For any $U \in \mathcal{C}_0$, the sequence $0 \to A(U) \to B(U) \to C(U)$ is exact, and $B \to C$ is a local epimorphism.

Besides the above method, there are also cases where exactness can be tested 'stalkwise'. Recall from the fourth lecture that we can define a *point of a topos* $\mathcal{E}$ as a geometric morphism

$$\mathrm{Set} \underset{x_*}{\overset{x^*}{\underset{\bot}{\leftrightarrows}}} \mathcal{E}$$

For $A \in \mathcal{E}_0$, we call x^*A the stalk of A at x, and often use the notation $A_x \coloneqq x^*A$.

The aforementioned map induces a geometric morphism $\mathrm{Ab} \to \mathrm{Ab}(\mathcal{E})$. The following definition will allow us to test exactness on the stalks

Definition 10. A topos $\mathcal{E}$ *has enough points* if for any morphism $f\colon A \to B$, it holds that if for any point x of $\mathcal{E}$, x^*f is an isomorphism, then f is an isomorphism.

Proposition 11. Suppose $\mathcal{E}$ is a topos that has enough points, and $A, B, C \in \mathrm{Ab}(\mathcal{E})_0$. Then the following are equivalent:

1. The sequence $0 \to A \to B \to C \to 0$ is exact.
2. For any point x of $\mathrm{Ab}(\mathcal{E})$, the sequence $0 \to A_x \to B_x \to C_x \to 0$ is exact.

Proof. This is part of the homework. □

For a topological space X, we have seen the category $\mathrm{Op}(X)$ of opens of X.

Proposition 12. Let $\mathcal{E} = \mathrm{Sh}(X, J) \coloneqq \mathrm{Sh}(\mathrm{Op}(X), J)$. Then $\mathcal{E}$ has enough points.

Recall from Lecture 4 that the sheaves on a topological space correspond to the étale spaces of that space, i.e. $\mathrm{Sh}(X) \simeq \mathrm{Et}(X)$. In the corresponding homework, we also saw that $\mathrm{Hom}_{\mathrm{Cat}}^{\mathrm{flat,cont}}(\mathcal{C}, \mathcal{E}) \simeq \mathrm{Hom}_{\mathrm{Topos}}(\mathcal{E}, \mathrm{Sh}(\mathcal{C}, J))$, and in particular that the points of $\mathrm{Sh}(X)$ correspond to the completely prime filters of opens in $\mathrm{Op}(X)$.

Example 13. We recall what we saw in Homework 4:

- Let $X = \{\cdot\}$ be the singleton space. Then $\mathrm{Sh}(X) \simeq \mathrm{Set}$, and $\mathrm{Hom}_{\mathrm{Topos}}(\mathrm{Set}, \mathrm{Sh}(X)) = \{\mathrm{id}_{\mathrm{Set}}\}$.
- Let $X = \{p, q\}$ be the 2-point space with discrete topology. Then $\mathrm{Hom}_{\mathrm{Topos}}(\mathrm{Set}, \mathrm{Sh}(X)) = \{f_p, f_q\}$, where f_p corresponds to the map $f_p' \colon \mathrm{Op}(X) \to \mathrm{Set}$ such that $f_p(U) = 1$ if $p \in U$ and 0 otherwise. The map f_q is defined symmetrically for the point q.
- Let $X = \{p, q\}$ be the 2-point space with the trivial topology. Then $\mathrm{Hom}_{\mathrm{Topos}}(\mathrm{Set}, \mathrm{Sh}(X))$ is a singleton set.

In general, it holds that the points of a topological space correspond to the points of $\mathrm{Sh}(X)$ when X is a *sober* space, i.e. every irreducible closed subset of X is the closure of a unique point. Note that in the example above, the first two spaces are sober, but the last is not.

3 Local sheaf cohomology

In the first section, we defined the i-th sheaf cohomology for abelian objects of a topos $\mathcal{E}$ (i.e. abelian valued sheaves), using the 'abelianization' of the global sections functor $\mathrm{Ab}(\Gamma)(-) = \mathrm{Hom}_{\mathrm{Ab}(\mathcal{E})}(*, -)$. We denoted the i-th sheaf cohomology of a sheaf $F \in \mathrm{Ab}(\mathcal{E})_0$ as $H^i(\mathcal{E}, F)$. For a general object $U \in \mathcal{E}$, we can define a more general sheaf cohomology over U, denoted $H^\bullet(U, F)$.

To do this, we first introduce the 'free abelian object' functor. As the name suggests, this is an adjoint of a forgetful functor.

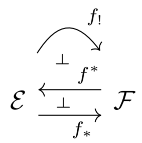
Definition 14. Let $\mathcal{E} = \mathrm{Sh}(\mathcal{C}, J)$ be a topos. Then we have the forgetful functor $i \colon \mathrm{Ab}(\mathcal{E}) \to \mathcal{E}$. This functor has a left adjoint $\mathbb{Z}[-] \colon \mathcal{E} \to \mathrm{Ab}(\mathcal{E})$, defined as follows: we see $\mathcal{F} \in \mathcal{E}_0$ as a presheaf, and then define $\mathbb{Z}[\mathcal{F}]$ pointwise by applying the free abelian group functor to the set $\mathcal{F}(U)$ for all $U \in \mathcal{C}$. After that, we apply the sheafification functor to get a sheaf again.

Definition 15. Let $\mathcal{E}$ be a topos, let $U \in \mathcal{E}_0$ and let $F \in \mathrm{Ab}(\mathcal{E})_0$. We define the *$i$-th sheaf cohomology over U* as the i-th right derived functor of the functor $\mathrm{Hom}_{\mathrm{Ab}(\mathcal{E})}(\mathbb{Z}[U], -)$, i.e. $H^i(U, F) = R^i\mathrm{Hom}_{\mathrm{Ab}(\mathcal{E})}(\mathbb{Z}[U], -)F$.

Note that taking $U = *$ in the definition above yields sheaf cohomology as in Definition 5.

We will look at the slice topos/over topos $\mathcal{E}/U$ and see an isomorphism between cohomologies there and the cohomology above. For this, we use that the geometric morphism $j \colon \mathrm{Ab}(\mathcal{E}) \to \mathcal{E}$ is essential.

Definition 16. Let $\mathcal{E}, \mathcal{F}$ be topoi. A geometric morphism $f \colon \mathcal{E} \to \mathcal{F}$ is called *essential* if f^* has a left-adjoint, denoted $f_!$, thus giving us the following diagram:



As with any geometric morphism, we have that an essential geometric morphism $f \colon \mathcal{E} \to \mathcal{F}$ induces an adjunction $\mathrm{Ab}(f_*) \colon \mathrm{Ab}(\mathcal{E}) \leftrightarrows \mathrm{Ab}(\mathcal{F}) : \mathrm{Ab}(f^*)$. By abstract nonsense, we can find the existence of a left-adjoint of $\mathrm{Ab}(f^*)$, which we call $\mathrm{Ab}(f_!)$. It should be noted that $\mathrm{Ab}(f_!)$ is not simply obtained by applying $f_!$ to the abelian group objects of $\mathcal{E}$, and that $\mathrm{Ab}(f_!)$ and $f_!$ may be very different maps. We often write $f_!$ instead of $\mathrm{Ab}(f_!)$ when it is clear from the context which of the two maps we mean.

Recall from the fourth lecture that for $\mathcal{E}$ a topos and any $U \in \mathcal{E}_0$, the slice category $\mathcal{E}/U$ is also a topos. We then saw that the inclusion map gave rise to a geometric morphism $j_U : \mathcal{E}/U \to \mathcal{E}$. This morphism happens to be essential, and the induced map $j_! : \mathrm{Ab}(\mathcal{E}/U) \to \mathrm{Ab}(\mathcal{E})$, *extension by zero functor*, is exact.

Proposition 17. The extension by zero functor is exact.

Proof. For a general proof, see Proposition 11.3.1 in Exposé IV of [2]. We will give a proof outline. For now, suppose $\mathcal{E} = \mathrm{PSh}(\mathcal{C})$, and suppose that $U \in \mathcal{C}_0$. We will look at $j_! : \mathrm{Ab}(\mathcal{E}/y(U)) \to \mathrm{Ab}(\mathcal{E})$. Firsly, note that we have an isomorphism $\mathcal{E}/y(U) \simeq \mathrm{PSh}(\mathcal{E}/U)$, which induces the isomorphism $\mathrm{Ab}(\mathcal{E}/y(U)) \simeq \mathrm{PSh}(\mathcal{E}/U; \mathrm{Ab})$. Then, by construction of $j_!$ (which we did not show in the notes, see for example the aforementioned reference), we have that for $A \in \mathrm{Ab}(\mathcal{E}/y(U))$, $j_!(A)(X) = \bigoplus_{u \in \mathrm{Hom}_{\mathcal{C}}(X,U)} A(X)$ for any $X \in \mathcal{C}_0$. This map is left-adjoint to j^*, thus right exact. So we only need to show that it preserves monomorphisms. The latter follows from the fact that limits of presheaves are computed object-wise.

For the general case, we have to apply the fact that we can always assume without loss of generality that U comes from an object of $\mathcal{C}$ (again, see the reference), and that the sheafification functor is exact. □

The following example illustrates why $j_!$ is often called the extension by zero functor.

Example 18. Let X be a topological space, $U \subseteq X$ an open subset of X and $N \in \mathrm{PSh}(X/U)$. Then for any open $V \subseteq X$, we have $j_!(N)(V) = \begin{cases} N(V) & \text{if } V \subseteq U \\ 0 & \text{otherwise} \end{cases}$. Note that this follows directly from the definition of $\mathrm{Op}(X)$ and $j_!$.

From the existence and exactness of $j_!$, and it being left-adjoint to j^*, we get an isomorphism between sheaf cohomologies.

Proposition 19. Let $\mathcal{E}$ be a topos, $U \in \mathcal{E}_0$ and $A \in \mathrm{Ab}(\mathcal{E})_0$. Then we have the following isomorphism:

$$H^n(U, A) \simeq H^n(\mathcal{E}/U, j^*(A)).$$

For the homework, see the next page!

4 Homework

Exercise 1

Prove Proposition 11. Hint: check out the definition of *filtered categories* and *filtered colimits*.

Exercise 2

Let Ω be the Sierpinski space, that is the topological space consisting of two points 0 and 1, where the opens subsets are $\emptyset, \{1\}$ and Ω.

(a) Show that the canonical morphism $p\colon \mathrm{Sh}(\Omega) \to \mathrm{Set}$ is essential, i.e. the functor $\mathrm{Const} = p^*\colon \mathrm{Set} \to \mathrm{Sh}(\Omega)$ has a left adjoint $p_!\colon \mathrm{Sh}(X) \to \mathrm{Set}$. (Note: p is the morphism as seen underneath Definition 1, where $\mathcal{E} = \mathrm{Sh}(\Omega)$).

(b) Show that the global sections functor $\Gamma = p_*\colon \mathrm{Sh}(\Omega) \to \mathrm{Set}$ also has a right adjoint $p^!\colon \mathrm{Set} \to \mathrm{Sh}(\Omega)$.

(c) Deduce from (b) that all cohomology groups $H^i(\Omega, F)$ with $i > 0$ and F an abelian sheaf, are trivial.

References

[1] *An Introduction to Sheaves on Grothendieck Topologies*. `https://webusers.imj-prg.fr/~pierre.schapira/LectNotes/SHV.pdf`. Accessed: 2025-03-24.

[2] Michael Artin, Alexander Grothendieck, and Jean-Louis Verdier. *Theorie de Topos et Cohomologie Etale des Schemas I, II, III.* Vol. 269, 270, 305. Lecture Notes in Mathematics. Springer, 1971.

[3] Saunders MacLane and Ieke Moerdijk. *Sheaves in geometry and logic: a first introduction to topos theory.* Universitext. Berlin: Springer, 1992. DOI: `10.1007/978-1-4612-0927-0`.