Homework 13

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Exercise 1. Recall that for a sheaf of groups $\mathcal{G}$, there is a sheaf of outer automorphisms $\underline{\mathrm{Out}}(\mathcal{G})$. A section $\phi \in \underline{\mathrm{Out}}(\mathcal{G})(U)$ is represented by a cover $\{U_\alpha\}_\alpha$ of U together with automorphisms $\phi_\alpha \in \mathrm{Aut}(\mathcal{G}|_{U_\alpha})$. They are locally compatible as outer automorphisms. This means that there are open covers $\{U^\xi_{\alpha\beta}\}_\xi$ of $U_{\alpha\beta}$, together with elements $\lambda^\xi_{\alpha\beta} \in \mathcal{G}(U^\xi_{\alpha\beta})$ such that $\phi_\alpha|_{U^\xi_{\alpha\beta}} = (\lambda^\xi_{\alpha\beta})_* \phi_\beta|_{U^\xi_{\alpha\beta}}$.

(a) (5 points) Fix some α, β. Prove that if the sheaf cohomology $H^1(U_{\alpha\beta}, Z(\mathcal{G}))$ vanishes, then these $\lambda^\xi_{\alpha\beta} \in \mathcal{G}(U^\xi_{\alpha\beta})$ may be replaced by a single $\lambda_{\alpha\beta} \in \mathcal{G}(U_{\alpha\beta})$.

(Hint: Remember the exact sequence $0 \to Z(\mathcal{G}) \to \mathcal{G} \to \underline{\mathrm{Aut}}(\mathcal{G}) \to \underline{\mathrm{Out}}(\mathcal{G}) \to 0$.)

(b) (4 points) Show that if $H^1(U, \mathcal{G}) = 0$ and $H^2(U, Z(\mathcal{G})) = 0$ then any such $\phi \in \underline{\mathrm{Out}}(\mathcal{G})(U)$ can be represented by an actual automorphism $\phi \in \underline{\mathrm{Aut}}(\mathcal{G})(U)$.

Exercise 2.

(a) (3 points) Prove that the stack of $\mathcal{G}$-torsors is a Gerbe.

(b) (3 points) What is its band?