Homework 13

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Exercise 1. Recall that for a sheaf of groups $\mathcal{G}$, there is a sheaf of outer automorphisms $\underline{\mathrm{Out}}(\mathcal{G})$. A section $\phi \in \underline{\mathrm{Out}}(\mathcal{G})(U)$ is represented by a cover $\{U_\alpha\}_\alpha$ of U together with automorphisms $\phi_\alpha \in \underline{\mathrm{Aut}}(\mathcal{G})(U_\alpha) = \mathrm{Aut}(\mathcal{G}|_{U_\alpha})$. They are locally compatible as outer automorphisms. This means that there are open covers $\{U^\xi_{\alpha\beta}\}_\xi$ of $U_{\alpha\beta}$, together with elements $\lambda^\xi_{\alpha\beta} \in \mathcal{G}(U^\xi_{\alpha\beta})$ such that $\phi_\alpha|_{U^\xi_{\alpha\beta}} = (\lambda^\xi_{\alpha\beta})_* \phi_\beta|_{U^\xi_{\alpha\beta}}$.

(a) (5 points) Fix some α, β. Prove that if the sheaf cohomology $H^1(U_{\alpha\beta}, Z(\mathcal{G}))$ vanishes, then these $\lambda^\xi_{\alpha\beta} \in \mathcal{G}(U^\xi_{\alpha\beta})$ may be replaced by a single $\lambda_{\alpha\beta} \in \mathcal{G}(U_{\alpha\beta})$.

(Hint: Remember the exact sequence $0 \to Z(\mathcal{G}) \to \mathcal{G} \to \underline{\mathrm{Aut}}(\mathcal{G}) \to \underline{\mathrm{Out}}(\mathcal{G}) \to 0$.)

(b) (4 points) Show that if $H^1(U, \mathcal{G}) = 0$ and $H^2(U, Z(\mathcal{G})) = 0$ then any such $\phi \in \underline{\mathrm{Out}}(\mathcal{G})(U)$ can be represented by an actual automorphism $\phi \in \underline{\mathrm{Aut}}(\mathcal{G})(U)$.

Solution 1.

(a) Split the exact sequence into

$$0 \to Z(\mathcal{G}) \to \mathcal{G} \to \mathcal{B} \to 0$$

and

$$0 \to \mathcal{B} \to \underline{\mathrm{Aut}}(\mathcal{G}) \to \underline{\mathrm{Out}}(\mathcal{G}) \to 0.$$

(+1pt) We obtain an exact sequence

$$Z(\mathcal{G})(U_{\alpha\beta}) \to \mathcal{G}(U_{\alpha\beta}) \to \mathcal{B}(U_{\alpha\beta}) \to H^1(U_{\alpha\beta}, Z(\mathcal{G})) \to \cdots$$

and hence a SES

$$0 \to Z(\mathcal{G})(U_{\alpha\beta}) \to \mathcal{G}(U_{\alpha\beta}) \to \mathcal{B}(U_{\alpha\beta}) \to 0$$

(+1pt) we also have

$$0 \to \mathcal{B}(U_{\alpha\beta}) \to \underline{\mathrm{Aut}}(\mathcal{G})(U_{\alpha\beta}) \to \underline{\mathrm{Out}}(\mathcal{G})(U_{\alpha\beta})$$

and hence an exact sequence

$$0 \to Z(\mathcal{G})(U_{\alpha\beta}) \to \mathcal{G}(U_{\alpha\beta}) \to \underline{\mathrm{Aut}}(\mathcal{G})(U_{\alpha\beta}) \xrightarrow{q} \underline{\mathrm{Out}}(\mathcal{G})(U_{\alpha\beta}).$$

(+1pt) We know that $\phi_\alpha|_{U_{\alpha\beta}}$ and $\phi_\beta|_{U_{\alpha\beta}}$ represent the same *outer* automorphism $\phi|_{U_{\alpha\beta}}$. We obtain $q(\phi_\alpha|_{U_{\alpha\beta}} \phi_\beta^{-1}|_{U_{\alpha\beta}}) = 1$ (where q is the quotient map) (+1pt). By the exactness of the above sequence this then lifts to some $\lambda_{\alpha\beta} \in \mathcal{G}(U_{\alpha\beta})$ such that $\phi_\alpha|_{U_{\alpha\beta}} \phi_\beta^{-1}|_{U_{\alpha\beta}} = (\lambda_{\alpha\beta})_*$ (+1pt).

(b) We split the sequence again (+1pt). We have a seven-term exact sequence

$$Z(\mathcal{G})(U) \to \mathcal{G}(U) \to \mathcal{B}(U) \to H^1(U, Z(\mathcal{G})) \to H^1(U, \mathcal{G}) \to H^1(U, \mathcal{B}) \to H^2(U, Z(\mathcal{G}))$$

hence $H^1(U, \mathcal{B}) = 0$ (+1pt). The 'long' exact sequence

$$\mathcal{B}(U) \to \underline{\mathrm{Aut}}(\mathcal{G})(U) \to \underline{\mathrm{Out}}(\mathcal{G})(U) \to H^1(U, \mathcal{B}) \to \cdots$$

gives us a short exact sequence

$$\mathcal{B}(U) \to \underline{\mathrm{Aut}}(\mathcal{G})(U) \to \underline{\mathrm{Out}}(\mathcal{G})(U) \to 0$$

(+1pt) hence $\underline{\mathrm{Aut}}(\mathcal{G})(U) \to \underline{\mathrm{Out}}(\mathcal{G})(U)$ is surjective. The conclusion follows. (+1pt)

Exercise 2.

(a) (3 points + 3 bonus points) Prove that the stack of $\mathcal{G}$-torsors is a Gerbe.

(b) (2 points + 1 bonus point) What is its band?

Solution 2.

(a)

$\mathcal{G}$-torsors has a global section given by $\mathcal{G}$ itself, which shows local non-emptiness (+1pt). For local connectedness, consider two $\mathcal{G}|_U$-torsors $\mathcal{S}, \mathcal{T}$. Let $x \in U$, then there is some open $V \ni x$ such that $\mathcal{S}, \mathcal{T}$ have sections on V (+1pt). In particular, $\mathcal{S}|_V \cong \mathcal{G}|_V \cong \mathcal{T}|_V$ (+1pt).

Now the next part I forgot about and hence was not intended to be part of the homework. However, since this was not clear by the phrasing of the question it is only fair to distribute bonus points to whomever decided to prove this.

We show that any morphism of $\mathcal{G}$-torsors $f : \mathcal{S} \to \mathcal{T}$ is an isomorphism. Since this is a map of sheaves it suffices to check this locally (+1 bonus pt). Let $x \in X$ and note that there is some neighborhood $U \ni x$ such that $\mathcal{S}(U), \mathcal{T}(U) \neq \emptyset$ (+1 bonus pt). In this case $f_U : \mathcal{S}(U) \to \mathcal{T}(U)$ is just a map of $\mathcal{G}(U)$-torsors in Set, and hence an isomorphism (+1 bonus pt).

(b) $\mathcal{G}$-torsors has a global section given by $\mathcal{G}$ (+1pt). This means that its band is just a sheaf of groups, namely $\underline{\mathrm{Aut}}_{\mathcal{G}\text{-torsors}}(\mathcal{G})$ (+1pt), which is $\mathcal{G}$ by standard arguments (+1 bonus pt).