Assignment 3: Seminar about Topos Theory
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Remark: All toposes on this exercise sheet are assumed to be *Grothendieck* toposes.

Exercise 1: Let $\mathcal{T}$ be a topos and $f\colon X \to Z$ a morphism in $\mathcal{T}$.

(a): Show that there exist $Y \in \mathcal{T}$ together with an epimorphism $\pi\colon X \twoheadrightarrow Y$ and a monomorphism $\iota\colon Y \hookrightarrow Z$ such that $f = \iota \circ \pi$.

(b): Show that between any two triples (Y, π, ι) and (Y', π', ι') with this property there exists a unique isomorphism $g\colon Y \to Y'$ such that $\pi' = g \circ \pi$ and $\iota = \iota' \circ g$.

(c): Deduce that any morphism in $\mathcal{T}$ which is a monomorphism and an epimorphism is an isomorphism.

Exercise 2: Let $\mathcal{E}$ be a category which has all pullbacks. Let $f\colon X \to Y$ be a morphism in $\mathcal{E}$. This morphism induces a pullback functor $f^*\colon \mathcal{E}/Y \to \mathcal{E}/X$. Indeed, to any $g\colon Z \to Y$ we can assign the pullback $X \times_Y Z \to X$ of g along f. Also, for any morphism $(Z \to Y) \to (Z' \to Y)$ in $\mathcal{E}/Y$ there is a natural morphism $(X \times_Y Z \to X) \to (X \times_Y Z' \to X)$ in $\mathcal{E}/X$.

(a): Assume in addition that $\mathcal{E}$ has a final object 1. Fix objects A and B in $\mathcal{E}$, and let $h\colon A \to 1$ be the unique arrow with the pullback functor $h^*\colon \mathcal{E}/1 \to \mathcal{E}/A$. Show that there is a natural one-to-one correspondence between the morphisms $A \to B$ in $\mathcal{E}$ and the morphisms $\mathrm{id}_A \to h^*(B \to 1)$ in $\mathcal{E}/A$.

Suppose now that $\mathcal{E}$ is a topos with initial object 0 and let $f\colon X \to Y$ be a morphism in $\mathcal{E}$. We saw in the lecture that both $\mathcal{E}/X$ and $\mathcal{E}/Y$ are toposes. Using this (and some additional properties of f^*) one can prove that f^* preserves finite colimits. You are allowed to use this fact for the next question.

(b): Suppose that there exists a morphism $X \to 0$. Show that $X \cong 0$.

Exercise 3: Indicate in the following table whether the given categories are toposes. You do not have to motivate your answer and you are allowed to use exercise 1 and 2 (the definition of a topos is also useful here). Let $\mathcal{C}$ be a small category.

Category:	$\varnothing$	$\{*\}$	Ab	Crings	CompactTop	Psh($\mathcal{C}$)
Topos (Y/N):						