Topos Theory: Sheaf Cohomology, Week 4 Homework

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Exercise 1

(a)

Let $(\mathcal{C}, J)$ be a site and $\mathcal{E}$ a topos, and let

$$p : \mathcal{E} \to \mathrm{PSh}(\mathcal{C}) \quad \text{and} \quad i : \mathrm{Sh}(\mathcal{C}, J) \to \mathrm{PSh}(\mathcal{C})$$

be geometric morphisms, with i being the usual embedding.

(i) Show that: p factors through i (as geometric morphisms) $\iff$ p_* factors through i_* (as functors).

(ii) Show that: p_* factors through i_* $\iff$ for every object C in $\mathcal{C}$ and covering sieve $S \in J(C)$, the inclusion $S \hookrightarrow \mathrm{y}_C$ induces an isomorphism $p^*(S) \xrightarrow{\sim} p^*(\mathrm{y}_C)$.

(b)

Let $(\mathcal{C}, J)$ be a site, $\mathcal{E}$ a topos, and $\mathrm{y} : \mathcal{C} \to \mathrm{PSh}(\mathcal{C})$ the Yoneda embedding (functor, not geometric morphism). Recall the following lemma:

Lemma 1. *For each flat functor $f : \mathcal{C} \to \mathcal{E}$,*[1] *there is a unique geometric morphism $p : \mathcal{E} \to \mathrm{PSh}(\mathcal{C})$ (up to natural isomorphism) with $f = p^* \circ \mathrm{y}$.*

In fact, this correspondence gives rise to an equivalence of categories

$$\mathrm{Hom}^{\mathrm{flat}}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{E}) \simeq \mathrm{Hom}_{\mathbf{Topos}}(\mathcal{E}, \mathrm{PSh}(\mathcal{C}))$$

We say $f : \mathcal{C} \to \mathcal{E}$ is *continuous* if it sends covering sieves to colimiting cocones.[2] Now, given the above, prove the following refinement of this lemma:

[1] If $\mathcal{C}$ has finite limits, f being flat means it preserves finite limits. However, for the purposes of this question, you do not need to know anything about what flatness is, besides it being some condition a functor (from a category to a topos) might satisfy.

[2] More precisely, a sieve $S \in J(C)$ may be viewed in $\mathcal{C}$ as a cocone of C over the diagram given by the forgetful functor $U : \mathrm{Elts}(S) \to \mathcal{C}$. Passing through f gives a cocone of $f(C)$ over the diagram $f \circ U$, and f being continuous means all such cocones are colimiting.

Lemma 2. *For each flat and continuous functor $f : \mathcal{C} \to \mathcal{E}$, there is a unique geometric morphism $q : \mathcal{E} \to \mathrm{Sh}(\mathcal{C}, J)$ (up to natural isomorphism) with $f = q^* \circ i^* \circ \mathrm{y}$ (where $i_* : \mathrm{PSh}(\mathcal{C}) \to \mathrm{Sh}(\mathcal{C}, J)$ is the sheafification functor).*

Bonus: show that this correspondence gives rise to an equivalence of categories

$$\mathrm{Hom}^{\mathrm{flat,cont}}_{\mathbf{Cat}}(\mathcal{C}, \mathcal{E}) \simeq \mathrm{Hom}_{\mathbf{Topos}}(\mathcal{E}, \mathrm{Sh}(\mathcal{C}, J))$$

(this is just checking details, I haven't done it myself yet and I don't think there is anything interesting here).

Exercise 2

(a)

Let $\mathrm{Sh}(X)$ be the topos of sheaves on a topological space X. Recall that a point of $\mathrm{Sh}(X)$ is a geometric morphism $\mathbf{Set} \to \mathrm{Sh}(X)$, and we may identify such points with flat and continuous functors $f : \mathrm{Op}(X) \to \mathbf{Set}$ (note: since $\mathrm{Op}(X)$ has finite limits, f being flat means it preserves finite limits).

(i) Show that such an f must map each U to either an empty or singleton set. Thus we may consider f an indicator function on $\mathrm{Op}(X)$, with corresponding set of opens $S_f = \{U \in \mathrm{Op}(X) \mid f(U) = 1\}$.

(ii) Show that S_f is a filter. That is: (1) $X \in S_f$; (2) $\emptyset \notin S_f$; (3) S_f is closed under finite intersections; (4) S_f is upwards closed (if $U \subseteq V$ and $U \in S_f$, then $V \in S_f$).

(iii) A filter $S \subseteq \mathrm{Op}(X)$ is *completely prime* if for any $U \in S$ and open cover $U = \bigcup_i U_i$, $U_i \in S$ for some i. Show that S_f is completely prime.

Bonus: show the converse. That is, if S is a completely prime filter of opens in $\mathrm{Op}(X)$, then $S = S_f$ for some flat and continuous $f : \mathrm{Op}(X) \to \mathbf{Set}$.

(b)

Describe explicitly the category $\mathrm{Hom}_{\mathbf{Topos}}(\mathbf{Set}, \mathrm{Sh}(X))$ of points of $\mathrm{Sh}(X)$ where

(i) X is the discrete 2-point space,

(ii) X is the indiscrete 2-point space,

(iii) X is the Sierpinski space (2 points, one closed, one open).

In each case, justify your answer.