Seminar Topos Theory: Sheaf Cohomology Homework
Abelian Categories

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Exercise 1

Let $\mathcal{A}$ be an abelian category, $A, B \in \mathrm{Ob}(\mathcal{A})$ and $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$. Show that the following are equivalent:

(i) f is a monomorphism;

(ii) for all $C \in \mathrm{Ob}(\mathcal{A})$ and all $g \in \mathrm{Hom}_{\mathcal{A}}(C, A)$, we have that if $f \circ g = 0$, then $g = 0$;

(iii) $\ker f = 0$.

You may use this and the analogous statements for epimorphisms in the next exercises.

Exercise 2: The First Isomorphism Theorem

Let $\mathcal{A}$ be an abelian category, $A, B \in \mathrm{Ob}(\mathcal{A})$ and $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$.

(a) Construct a map $h\colon \mathrm{coim} f \to \mathrm{im}\, f$ such that the following diagram commutes:

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ \downarrow & & \uparrow \\ \mathrm{coim} f & \xrightarrow[h]{} & \mathrm{im} f \end{array}$$

(b) Show that h is a monomorphism. (Hint: Show that $\mathrm{coim} f \xrightarrow{h} \mathrm{im}\, f \to B$ is a monomorphism.)

(c) Show that any morphism which is both a monomorphism and an epimorphism in $\mathcal{A}$, is an isomorphism.

You can use an argument very similar to (b) to show that h is an epimorphism. Now by Exercise (c), h is an isomorphism.

Exercise 3

Let $\mathcal{A}$ be an abelian category. Construct a preadditive category $\mathcal{C}$ such that $[\mathcal{C}, \mathcal{A}]^{\mathrm{add}}$ is the category of cochain complexes in $\mathcal{A}$. You do not need to prove this, just define the objects of $\mathcal{C}$, the hom-sets between the objects, the group structure on the hom-sets and the composition. Recall that a cochain complex in $\mathcal{A}$ is a sequence of objects and maps

$$\ldots \xrightarrow{\partial_{-1}} A_{-1} \xrightarrow{\partial_0} A_0 \xrightarrow{\partial_1} A_1 \xrightarrow{\partial_2} \ldots$$

in $\mathcal{A}$ such that $\partial_{n+1} \circ \partial_n = 0$ for every $n \in \mathbb{Z}$.

Bonus Exercise

Let $\mathcal{A}$ be an abelian category where every short exact sequence splits, i.e. if $0 \to A \to B \to C \to 0$ is a short exact sequence in $\mathcal{A}$, then there exists an isomorphism $\varphi\colon B \to A \oplus C$ such that

$$\begin{array}{ccccccccc} 0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \Big\downarrow{\scriptstyle \varphi} & & \Big\downarrow{\scriptstyle \mathrm{id}} & & \\ 0 & \longrightarrow & A & \xrightarrow[i]{} & A \oplus C & \xrightarrow[p]{} & C & \longrightarrow & 0 \end{array}$$

commutes, where i and p are the canonical maps. Show that every object of $\mathcal{A}$ is injective. You may use that for any morphism $A \xrightarrow{f} B$ in $\mathcal{A}$, both $\ker(f) \to A \xrightarrow{f} B$ and $A \xrightarrow{f} B \to \operatorname{coker}(f)$ are exact.