Seminar Topos Theory: Homological Algebra II

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Last week we saw that geometric morphisms of toposes induce adjunctions between the categories of abelian group objects in the toposes:

Proposition 1. *Let $F : \mathcal{C} \to \mathcal{D}$ be a geometric morphism of toposes. Then there are induced adjoint additive functors*

$$\mathrm{Ab}(\mathcal{C}) \underset{\mathrm{Ab}(F_*)}{\overset{\mathrm{Ab}(F^*)}{\underset{\longrightarrow}{\overset{\longleftarrow}{\bot}}}} \mathrm{Ab}(\mathcal{D}).$$

In particular, we saw that the induced global sections functor $\mathrm{Ab}(\Gamma) : \mathrm{Ab}(\mathcal{C}, \mathcal{J}) \to \mathbf{Ab}$ is a right adjoint, so it is left exact. The content in this lecture will be mostly detached from the context of toposes, so it is good to keep this example in mind.

Motivation: Given a left exact functor $F : \mathcal{A} \to \mathcal{B}$ between abelian categories with $\mathcal{A}$ having enough injectives, and a short exact sequence

$$0 \to A \to B \to C \to 0$$

in $\mathcal{A}$, can we measure the failure of F to preserve exactness of the sequence on the right? In other words, '*how far away from being right exact is F?*' We can try to answer this question by extending the sequence

$$0 \to FA \to FB \to FC$$

to the right. We will construct functors RF^i, called the *right derived functors* of F, that we can apply to the sequence to obtain

$$RF^i(A) \to RF^i(B) \to RF^i(C)$$

for each i. We will attach these 'chunks' together to obtain a long exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & FA & \longrightarrow & FB & \longrightarrow & FC \\ & & & \overset{\partial}{\swarrow} & & & \\ RF^1A & \longrightarrow & RF^1B & \longrightarrow & RF^1C & & \\ & & & \overset{\partial}{\swarrow} & & & \\ RF^2A & \longrightarrow & RF^2B & \longrightarrow & RF^2C & \longrightarrow & \dots \end{array}$$

The answer to our question is now encoded in this long exact sequence: the longer the sequence takes to vanish, the 'less F preserves exactness of our original short exact sequence'.

The recipe to compute R^iFA is vaguely:

1. Construct a long exact sequence of injective objects out of each $A \in \mathcal{A}$:

$$0 \to A \to I^0 \to I^1 \to \dots.$$

2. Apply the functor F to the sequence.

3. Compute the *ith cohomology* of the complex $0 \to FI^0 \to FI^1 \to \dots$, this measures how far away from being exact the complex is at index i.

Example 2. Let X be a topological space and $f : F \to G \in \mathrm{Ab}(X)$ an epimorphism of sheaves of abelian groups. The map f is an epimorphism if and only if the induced map on all stalks is an epimorphism. The global sections functor Γ preserves this epimorphism iff $\Gamma(f) : F(X) \to G(X)$ is surjective. So the derived functors of Γ will measure the failure of the surjectivity on stalks to assemble into surjectivity on the global sections.

Definition 3 (Injective resolutions)**.** Let $A \in \mathcal{A}$ be an object in an abelian category. A *resolution* of A is a long exact sequence

$$0 \to A \to I^0 \to I^1 \to I^2 \to \dots.$$

We'll often write $0 \to A \to I^\bullet$. If each I^i is an injective object, then $0 \to A \to I^\bullet$ is called an *injective resolution*. Dualise to obtain *projective resolutions*.

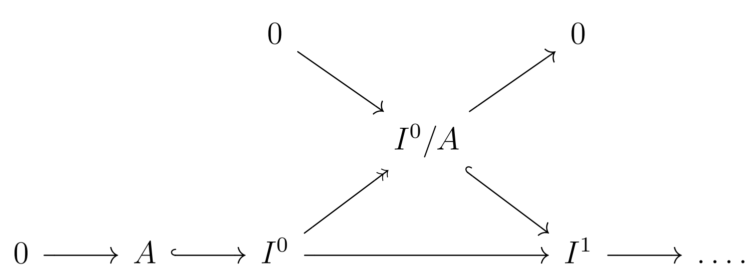
Example 4. In the category of abelian groups, the sequence

$$0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$$

is an injective resolution of $\mathbb{Z}$. Indeed the sequence is exact, $\mathbb{Q}$ is divisible, and quotients of divisible groups are divisible.

Proposition 5 (Splicing)**.** *Let $\mathcal{A}$ be an abelian category with enough injectives. Then for every object A there is an injective resolution $0 \to A \to I^\bullet$.*

Proof. Since $\mathcal{A}$ has enough injectives, we can pick $0 \to A \hookrightarrow I^0$. We now pick an injective object I^1 that the quotient I^0/A injects into:



We have exactness at I^0, because the kernel of the map $I^0 \to I^1$ is the kernel of the map $I^0 \to I^0/A$. Proceed inductively to complete the proof (the next step is to consider the cokernel of $I^0/A \to I^1$). □

Remark 6. Note that the above construction involves a choice of injective object at each step, so the resulting injective resolution is not canonical, and indeed there may be many injective resolutions for a given object. For example, the zero group $0 \in \mathbf{Ab}$ is injective, so $0 \to 0$ is an injective resolution. But so is $0 \to \mathbb{Q} \to \mathbb{Q} \to 0$.

Remark 7. In general, quotients of injective objects are not injective. In the case of abelian groups, the injective objects are the divisible groups, and quotients of these *are* divisible, so the splicing algorithm above trivialises.

Definition 8 (Cohomology)**.** Let $C^\bullet \in \mathrm{Ch}(\mathcal{A})$ be a chain complex for an abelian category $\mathcal{A}$, denote the chain maps by $d^i : C^i \to C^{i+1}$. The *ith cohomology of* $C^\bullet$ is defined by

$$H^i(C^\bullet) := \ker d^i / \operatorname{im} d^{i-1}.$$

Given a map of chain complexes $C^\bullet \to D^\bullet$, there are induced maps $H^i(C^\bullet) \to H^i(D^\bullet)$ that turn H^i into a functor $H^i : \mathrm{Ch}(\mathcal{A}) \to \mathcal{A}$.

Example 9. Consider the chain complex $C^\bullet$ given by

$$0 \to \mathbb{Z} \xrightarrow{\cdot 4} \mathbb{Z} \to \mathbb{Z}/2 \to 0,$$

where we start indexing at 0. Then

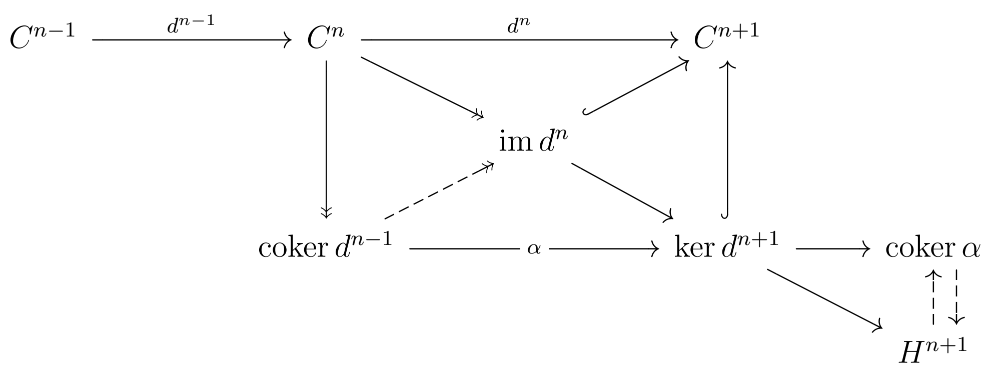
$$H^2(C^\bullet) = \ker(\mathbb{Z} \to \mathbb{Z}/2) / \operatorname{im}(\mathbb{Z} \to \mathbb{Z}) = 2\mathbb{Z}/4\mathbb{Z} = \mathbb{Z}/2.$$

There is another way to compute cohomology, which we will use to get the connecting morphism in the long exact sequence of cohomology later.

Lemma 10 (TIE Fighter Lemma)**.** *Let $C^\bullet$ be a chain complex. Then there are induced maps $\alpha_n : \operatorname{coker} d^{n-1} \to \ker d^{n+1}$, and we recover cohomology by natural isomorphisms*

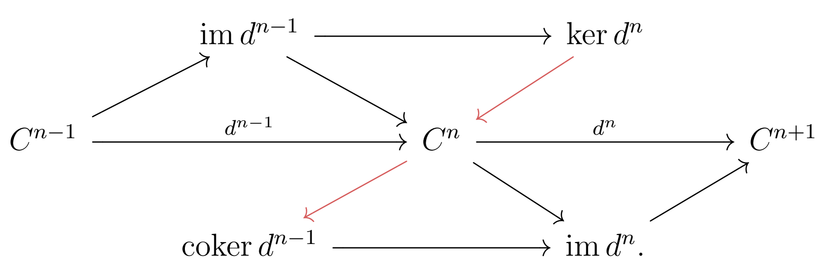
$$H^{n+1}(C^\bullet) \cong \operatorname{coker} \alpha_n, H^n(C^\bullet) \cong \ker \alpha_n.$$

Proof. For the cokernel, stare at the diagram



for a while. Convince yourself that there is an induced epimorphism $\operatorname{coker} d^{n-1} \to \operatorname{im} d^n$ ($C^{n-1} \to C^n \to C^{n+1}$ is zero and $\operatorname{im} d^n \to C^{n+1}$ is a mono so $C^{n-1} \to \operatorname{im} d^n$ is zero, so $C^n \to \operatorname{im} d^n$ factors via $\operatorname{coker} d^{n-1}$) and so that we get a map α. After that it's a direct check that $H^{n+1}(C^\bullet)$ and $\operatorname{coker} \alpha$ satisfy each other's universal properties.

That $\ker \alpha_n \cong H^n(C^\bullet)$ is a bit tricker to see. The following proof is taken from [1, p. 178]. First notice that $\ker \alpha_n = \ker(\operatorname{coker} d^{n-1} \to \operatorname{im} d^n)$, since $\operatorname{im} d^n \to \ker d^{n+1}$ is a monomorphism. We consider the TIE-fighter-looking diagram



The composition $u\colon \ker d^n \to \operatorname{coker} d^{n-1}$, shown in red, has cokernel $\operatorname{im} d^n$ (direct check since $\operatorname{im} d^n \cong \operatorname{coim} d^n$ is the cokernel of $\ker d^n \to C^n$). So the image of u is $\ker(\operatorname{coker} d^{n-1} \to \operatorname{im} d^n)$. Similary, $\operatorname{im} d^{n-1}$ is the kernel of u, so the coimage of u is $\operatorname{coker}(\operatorname{im} d^{n-1} \to \ker d^n)$, i.e. $H^n(C^\bullet)$.

□

Definition 11 (Right derived functors)**.** Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories, and suppose that $\mathcal{A}$ has enough injectives. For each object $A \in \mathcal{A}$, choose an injective resolution $0 \to A \to I^\bullet$. For each $i \in \mathbb{Z}_{\geq 0}$ define the *ith right derived functor of* F at A to be

$$R^i FA := H^i(FI^\bullet).$$

Note that the kernel of $FI^0 \to FI^1$ is FA because F is left exact. To make sense of the zeroeth cohomology, we consider the complex $0 \to FI^0 \to FI^1 \to \dots$, so $R^0FA = FA$.

We have not yet justified that the above definition makes sense on objects, because the choice of injective resolution for A is not unique. This is part of the homework for this week. We also have not defined what these functors do on morphisms! We will do that next. But we can first see an example of what happens on objects.

Example 12. From the injective resolution $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ of $\mathbb{Z}$ and the left exact functor $F = \operatorname{Hom}(\mathbb{Z}/2, -)$ we see that $\operatorname{Hom}(\mathbb{Z}/2, \mathbb{Z}) = 0$, $\operatorname{Hom}(\mathbb{Z}/2, \mathbb{Q}) = 0$, and $\operatorname{Hom}(\mathbb{Z}/2, \mathbb{Z}) = \mathbb{Z}/2$. So we have the chain complex

$$0 \to \mathbb{Z}/2 \to 0$$

and so $R^1F\mathbb{Z} = \mathbb{Z}/2$. This derived functor has a name, it is called $\operatorname{Ext}^1(\mathbb{Z}/2, -)$.

Theorem 13 (Baby Comparison Theorem, [2, Thm 2.3.7])**.** *Let $B \to J^\bullet$ be an injective resolution and $f' : A \to B$ be a map in an abelian category $\mathcal{A}$. For any resolution $A \to I^\bullet$ there exists a chain map $f : I^\bullet \to J^\bullet$ lifting f', i.e. a collection of maps $f^i : I^i \to J^i$ such that the diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & I^0 & \longrightarrow & I^1 & \longrightarrow & \dots \\
 & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f^0} & & \downarrow{\scriptstyle f^1} & & \\
0 & \longrightarrow & B & \longrightarrow & J^0 & \longrightarrow & J^1 & \longrightarrow & \dots
\end{array}$$

commutes.

Proof. Weibel gives a proof for the projective case in [2, Thm 2.2.6], which we dualise here. For the base case we can factor the composition $A \to J^0$ along the injection $A \to I^0$ because J^0 is injective. For the inductive step, suppose we have $f^n : I^n \to J^n$. The diagram

$$\begin{array}{ccccc}
I^{n-1} & \longrightarrow & \ker d_I^n & \longrightarrow & I^n \\
\downarrow{\scriptstyle f^{n-1}} & & \downarrow & & \downarrow{\scriptstyle f^n} \\
J^{n-1} & \longrightarrow & \ker d_J^n & \longrightarrow & J^n
\end{array}$$

commutes because by exactness $\ker d_I^n = \operatorname{im} d_I^{n-1}$ and the same for J, so we get an induced map $I^n / \ker d_I^n \to J^n / \ker d_J^n$ making the diagram

$$\begin{array}{ccccccc}
\ker d_I^n & \longrightarrow & I^n & \longrightarrow & I^n / \ker d_I^n & \longrightarrow & I^{n+1} \\
\downarrow & & \downarrow{\scriptstyle f^n} & & \downarrow & & \\
\ker d_J^n & \longrightarrow & J^n & \longrightarrow & J^n / \ker d_J^n & \longrightarrow & J^{n+1}
\end{array}$$

commute. But $I^n/\ker d_I^n$ is just the image of d_I^n, so the map $I^n/\ker d_I^n \to I^{n+1}$ is an injection, so we can again use the injectivity of J^{n+1} to conclude. $\square$

Now that we can extend morphisms of objects in $\mathcal{A}$ to morphisms of their injective resolutions, we can define what RF^i does on morphisms. But again, this is not apriori well defined. The homework takes care of this.

Next, a lemma that says we can construct injective resolutions that extend short exact sequences of objects to short exact sequences of complexes.

Lemma 14 (Horseshoe Lemma, [2, Lemma 2.2.8])**.** *Let $0 \to A \to B \to C \to 0$ be a short exact sequence in an abelian category $\mathcal{A}$. Suppose there are injective resolutions $0 \to A \to A^\bullet$ and $0 \to C \to C^\bullet$. Then $A \oplus C$ is an injective resolution of B and the resolutions assemble into a commutative diagram*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^0 & \longrightarrow & A^0 \oplus C^0 & \longrightarrow & C^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A^1 & \longrightarrow & A^1 \oplus C^1 & \longrightarrow & C^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \dots & & \dots & & \dots . & &
\end{array}$$

where the rows are exact.

Theorem 15 (Long exact sequence of cohomology, [2, Thm 1.3.1])**.** *Let $0 \to A^\bullet \to B^\bullet \to C^\bullet \to 0$ be a short exact sequence of chain complexes in $\mathrm{Ch}\,\mathcal{A}$, for $\mathcal{A}$ an abelian category. Then there is a natural long exact sequence*

$$\cdots \to H^i(A^\bullet) \to H^i(B^\bullet) \to H^i(C^\bullet) \xrightarrow{\partial} H^{i+1}(A^\bullet) \to H^{i+1}(B^\bullet) \to H^{i+1}(C^\bullet) \to \dots$$

in $\mathcal{A}$.

Proof. From the snake lemma the rows in the diagram

$$\begin{array}{ccccccccc}
 & & \operatorname{coker} d_A^{n-1} & \longrightarrow & \operatorname{coker} d_B^{n-1} & \longrightarrow & \operatorname{coker} d_C^{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker d_A^{n+1} & \longrightarrow & \ker d_B^{n+1} & \longrightarrow & \ker d_C^{n+1} & &
\end{array}$$

are exact. The kernels of the downward maps are cohomology in the nth degree, and the cokernels are cohomology in the $n+1$th degree, by lemma 10. So applying the snake lemma again gives the connecting morphism. $\square$

We can finally get back to our goal. Given $F : \mathcal{A} \to \mathcal{B}$ left exact, we can compute the derived functors at A, B and C using the resolutions from the Horseshoe Lemma. Applying F to the resolutions gives the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & FA^0 & \longrightarrow & FA^0 \oplus FC^0 & \longrightarrow & FC^0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & FA^1 & \longrightarrow & FA^1 \oplus FC^1 & \longrightarrow & FC^1 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \cdots & & \cdots & & \cdots, & &
\end{array}$$

where the rows are still exact because additive functors preserve split exact sequences. So this is a short exact sequence $0 \to FA^\bullet \to F(A^\bullet \oplus B^\bullet) \to FC^\bullet \to 0$ of chain complexes. We are now done by computing the long exact sequence of cohomology of this sequence.

Example 16. The short exact sequence $0 \to \mathbb{Z} \to \mathbb{Q} \to \mathbb{Q}/\mathbb{Z} \to 0$ is an injective resolution of $\mathbb{Z}$. The sequence $0 \to \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \to 0$ is an injective resolution of $\mathbb{Q}/\mathbb{Z}$.

We get the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Q} & \longrightarrow & \mathbb{Q} \oplus \mathbb{Q}/\mathbb{Z} & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & \mathbb{Q}/\mathbb{Z} & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

Applying the functor $\mathrm{Hom}(\mathbb{Z}/2, -)$ to the bottom two rows we get the diagram and computing cohomology, we get

$$\begin{array}{ccccccccc}
0 & \longrightarrow & 0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & \mathbb{Z}/2 & \longrightarrow & 0 & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0. & &
\end{array}$$

Now we compute cohomology on the columns, obtaining the sequence

$$0 \to 0 \to \mathbb{Z}/2 \xrightarrow{\partial} \mathbb{Z}/2 \to 0.$$

Notice how complicated this was in the middle. We could also have computed $\mathrm{Ext}^1(\mathbb{Z}/2, -)$ individually for each element in the sequence by picking an easier resolution for $\mathbb{Q}$. Then the fact that the derived functors assemble into a long exact sequence forces ∂ to be an isomorphism, there is only one, so the long exact sequence is determined.

Definition 17 (Sheaf cohomology)**.** Given a topos $\mathcal{T}$, the category of abelian group objects $\mathrm{Ab}(\mathcal{T})$ has enough injectives. Thus we can define the derived functors of the global sections functor

$$\mathrm{Ab}(\Gamma) : \mathrm{Ab}(\mathcal{T}) \to \mathbf{Ab}.$$

Given $F \in \mathcal{T}$, we call $R^i \,\mathrm{Ab}(\Gamma)F$ the *ith sheaf cohomology* of F.

Definition 18. Let $C, D \in \mathrm{Ch}(\mathcal{A})$ be complexes over an abelian category $\mathcal{A}$. Two morphisms of chain complexes $f, g : C^\bullet \to D^\bullet$ are called homotopic if there are maps $s^n : C^n \to D^{n-1}$ such that

$$f - g = ds + sd.$$

Spelled out, that means that for each n, we have $f^n - g^n = d^{n-1} \circ s^n + s^{n+1} \circ d^n$. By setting $g = 0$ we can define what it means for a map to be nullhomotopic. A *chain homotopy equivalence* is a pair of maps $a : C^\bullet \to D^\bullet$, $b : D^\bullet \to C^\bullet$ such that the compositions $a \circ b$ and $b \circ a$ are both homotopic to the identity.

Problem 1 (Right derived functors are well defined on morphisms)**.**

(a) Show that if two maps $f, g : C^\bullet \to D^\bullet$ are homotopic, they induce the same maps on cohomology.

(b) Suppose $f : C^\bullet$ and $D^\bullet$ is a chain homotopy equivalence. Show that f induces isomorphisms on cohomology on all degrees.

(c) Show that the chain map constructed in the Comparison Theorem is unique up to chain homotopy.

(d) Deduce that the derived functors R^iF are well defined for $F : \mathcal{A} \to \mathcal{B}$ left exact with $\mathcal{A}$ having enough injectives.

Problem 2 (Bonus)**.** Give an example of an abelian category $\mathcal{A}$, an injective object I and a monomorphism $i : A \to I$ such that $\operatorname{coker} i = I/A$ is not injective. *Hint 1: you won't find examples for modules over a PID. Hint 2: Baer's criterion may be helpful to prove that modules are or aren't injective.*

References

[1] M. Kashiwara and P. Schapira. *Categories and Sheaves.* Grundlehren der mathematischen Wissenschaften. Springer Berlin Heidelberg, 2005.

[2] Charles A Weibel. *An introduction to homological algebra.* Number 38. Cambridge university press, 1994.