Homework 9

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Group Homology: Given a group G, we let $\mathbb{Z}[G]$ denote the free abelian group on the underlying set of G. Multiplication on G extends to multiplication on $\mathbb{Z}[G]$, and we call this the group ring on G. The morphism $\epsilon : \mathbb{Z}[G] \to \mathbb{Z}$ which sends any $g \in G$ to 1 induces a functor $\epsilon^* : Ab \to \mathbb{Z}[G]\text{-}Mod$, and we have the following adjunction: $(\ )_G : \mathbb{Z}[G]\text{-}Mod \to Ab \dashv \epsilon^*$, where for any $\mathbb{Z}[G]$-module A, we have $(A)_G = \mathbb{Z} \otimes_{\mathbb{Z}[G]} A$.

Let G be a group and A be a $\mathbb{Z}[G]$-module. We define the group homology of G with coefficients in A (denoted by $H_*(G; A)$) as follows: $H_i(G; A) := L_i(\ )_G(A)$.
Note that $H_*(G; \mathbb{Z})$ is isomorphic to the homology of the chain complex we get from the nerve of the category BG (viewing G as a one object category).

Homology of Cyclic Groups

1. Consider $G = \mathbb{Z}/n\mathbb{Z} = \{\omega^i : 0 \leq i \leq n-1\}$. In $\mathbb{Z}/n\mathbb{Z}$, we consider the element $T = \sum_{i=0}^{n-1} \omega^i$ and $N = \omega - 1$.

   (a) Show that the complex

   $$... \longrightarrow \mathbb{Z}[G] \xrightarrow{\times T} \mathbb{Z}[G] \xrightarrow{\times N} ... \longrightarrow \mathbb{Z}[G] \xrightarrow{\times T} \mathbb{Z}[G] \xrightarrow{\times N} \mathbb{Z}[G]$$

   gives a free resolution of $\mathbb{Z}$ as a trivial $\mathbb{Z}[G]$-module.

   (b) Calculate $H_k(G; \mathbb{Z})$ for all $k \geq 0$.

   (c) (Bonus Exercise) Calculate $H_*(\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}/m\mathbb{Z}; \mathbb{Z})$. You can assume the Künneth Formula for group homology.