Notes Lecture 8

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March 28, 2025

1 Goal

Finally we have sheaf cohomology, does it relate to other theories?
$---->$ YES
De Rham cohomology
Singular cohomology
We will develop tools to conclude this (flabby, soft, fine, F-acyclic).

2 Acyclic resolutions

Injective resolutions are pretty nice in theory but they are hard to come by in the wild. The following are more common.

Definition. *The i'th sheaf cohomology of an object $\mathcal{F}$ in a topos $\mathcal{E}$ is the right derived functor of $\Gamma(-,\mathcal{F})$ i.e.*

$$H^i(\mathcal{E},\mathcal{F}) := R^i\Gamma(-,\mathcal{F}).$$

We will only talk about the topos $\mathrm{Sh}(X) := \mathrm{Sh}(\mathrm{Op}(X))$ today.

Definition. *Let $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories. We call an object A of $\mathcal{A}$ F-acyclic if*

$$R^iF(A) = 0$$

for all $i > 0$.

It is clear that injective objects are F-acyclic for every F.

Lemma. *Suppose $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor of abelian categories, A an object of $\mathcal{A}$ and*

$$0 \longrightarrow A \longrightarrow C^0 \longrightarrow C^1 \longrightarrow \cdots$$

be a resolution of A where all the objects C^i are F-acyclic. Then

$$R^iF(A) \cong H^i(FC^\bullet),$$

for each i.

Proof. (OPTIONAL) The case for $i = 0$ is clear by left exactness of F and $R^0F \cong F$. Define $B := \mathrm{im}(C^0 \to C^1) \cong \ker(C^1 \to C^2)$. Now we get an exact sequence

$$0 \longrightarrow A \longrightarrow C^0 \longrightarrow B \longrightarrow 0$$

and a long exact sequence

$$0 \longrightarrow B \longrightarrow C^1 \longrightarrow C^2 \longrightarrow \cdots$$

Then by the short exact sequence, we obtain a long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & R^0F(A) \arrow[r] & R^0F(C^0) \arrow[r] & R^0F(B) \arrow[dll] \\
& R^1F(A) \arrow[r] & R^1F(C^0) \arrow[r] & R^1F(B) \arrow[dll] \\
& R^2F(A) \arrow[r] & R^2F(C^0) \arrow[r] & R^2F(B) \arrow[dll] \\
& \dots & &
\end{tikzcd}$$

and since C^0 is F-acyclic, we get an exact sequence

$$\begin{tikzcd}
0 \arrow[r] & F(A) \arrow[r] & C^0 \arrow[r] & F(B) \arrow[dll] \\
& R^1F(A) \arrow[r] & 0 \arrow[r] & R^1F(B) \arrow[dll] \\
& R^2F(A) \arrow[r] & 0 \arrow[r] & R^2F(B) \arrow[dll] \\
& \dots & &
\end{tikzcd}$$

meaning that we get an isomorphism $R^iF(B) \cong R^{i+1}F(A)$ for each $i > 0$. Then by left exactness of F, the long exact sequence $0 \to B \to C^1 \to \dots$ gives an exact sequence

$$0 \longrightarrow F(B) \longrightarrow F(C^1) \longrightarrow F(C^2)$$

because F is left exact. Now it follows that

$$\begin{aligned} R^1F(A) &\cong \frac{F(B)}{F(C^0)} \\ &\cong \frac{\ker(F(C^1) \to F(C^2))}{\operatorname{im}(F(C^0) \to F(C^1))} \\ &\cong H^1(F(C^\bullet)) \end{aligned}$$

giving us the result for $i = 1$. Now for $i > 1$, an induction arguement to the long exact sequence $0 \to B \to C^1 \to \dots$ gives

$$\begin{aligned} R^iF(A) &\cong R^{i-1}(F(B)) \\ &\cong H^{i-1}(F(C^1) \to F(C^2) \to \dots) \\ &= H^i(F(C^0) \to F(C^1) \to \dots) \\ &= H^i(F(C^\bullet)). \end{aligned}$$

□

3 Flabby sheaves and singular cohomology

Definition. *A sheaf $\mathcal{F}$ on a topological space X is called flabby if for each inclusion of opens $U \subseteq V$, the restriction $\mathcal{F}(V) \to \mathcal{F}(U)$ is surjective.*

Lemma. *Suppose $\mathcal{F}$ is a flabby sheaf on X. Then $\mathcal{F}$ is $H^0(U, -)$-acyclic for any open $U \subseteq X$. (This just means $H^i(U, \mathcal{F}) = 0$ for each $i > 0$).*

Proof. Choose an injection $\mathcal{F} \hookrightarrow \mathcal{I}$ into an injective sheaf $\mathcal{I}$. Then these fit in an exact sequence

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{I} \longrightarrow \mathcal{H} \longrightarrow 0$$

where we may choose $\mathcal{H}$ as the quotient of $\mathcal{F}$ and $\mathcal{I}$. Since $\mathcal{I}$ is injective it is flabby and thus, $\mathcal{H}$ is flabby too (homework) and moreover

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{I}(U) \longrightarrow \mathcal{H}(U) \longrightarrow 0$$

is exact. Now $\mathcal{I}$ being injective implies $H^i(U, \mathcal{I}) = 0$ and thus by the long exact sequence on cohomology and an induction arguement show the result. □

Recall that for a space X and an open U, we have a cochain complex

$$0 \longrightarrow \operatorname{Hom}(C^0(U), A) \longrightarrow \operatorname{Hom}(C^1(U), A) \longrightarrow \cdots$$

where $C^i(U)$ is the free abelian group generated by maps $\Delta^i \to U$. In the latter, we define $C^\bullet(X, A) := \operatorname{Hom}(C^\bullet(U), A)$ Then singular cohomology is defined as

$$H^i_{\text{sing}}(U, A) := \frac{\ker \partial^i}{\operatorname{im} \partial^{i-1}}$$

.
The following theorem compares sheaf cohomology with singular cohomology. It is worth noting that the condition of hereditary paracompactness is not needed for the isomorphism, but it simplifies the proof vastly. If one wants to do it without hereditary paracompactness one needs the following tools:
-model categories
-godement resolution
-hypersheaves

Theorem. *If X is locally contractible and hereditarily paracompact then for every abelian group A, $H^i_{\text{sing}}(X, A) \cong H^i(X, \underline{A})$.*

Proof. -The assignment $U \mapsto C^i(U, A)$ defines a complex of presheaves $C^\bullet(U, A)$. Let $V^\bullet$ be the cocomplex of locally vanishing cochains. Then the sheaffification of $C^\bullet(-, A)^\# \simeq C/V(A)^\bullet$.
-The complex $C^\bullet(U, A)$ is exact for every contractible U. Thus each $C^i(-, A)$ is flabby. (here we use hereditary paracompactness)
-Thus we have a flabby resolution

$$0 \longrightarrow \underline{A} \longrightarrow C/V(A)^0 \longrightarrow C/V(A)^1 \longrightarrow \cdots$$

because all the $C^i(-, A)$ being flabby means $C/V(A)^i$ are flabby as well.
-One has a quasi-isomorphism

$$C^\bullet(X, A) \to (C/V)(A)^\bullet(X).$$

Putting it all together:

$$\begin{aligned} H^i_{\text{sing}}(X, A) &\cong H^i(C^\bullet(X, A)) \\ &\cong H^i((C/V)(A)^\bullet(X)) \\ &\cong H^i(\Gamma(X, (C/V)(A)^\bullet)) \\ &\cong R^i\Gamma(X, \underline{A}) \\ &\cong H^i(X, \underline{A}). \end{aligned}$$

□

4 Soft sheaves and de Rham cohomology

Definition. *A sheaf of abelian groups $\mathcal{F}$ on a topological X is called soft if for every closed $Z \subseteq X$, the restriction map $\mathcal{F}(X) \to \mathcal{F}|_Z(Z)$ is surjective.*

Proposition. *Let X be paracompact Hausdorff, Z closed in X and $\mathcal{F}$ a sheaf on X. Then we have an isomorphism*

$$(-)|_Z : \text{colim}_{U \supseteq Z,\ open} \mathcal{F}(U) \to \mathcal{F}|_Z(Z)$$

Proof. Omitted because it is ugly. □

Corollary. *Every flabby sheaf on a paracompact Hausdorff space is soft.*

Proposition. *A soft sheaf on a paracompact Hausdorff space X is $H^0(X,-)$-acyclic.*

Proof. Similar to showing flabby means $H^0(X,-)$-acyclic. $\square$

Lemma. *Let X be paracompact Hausdorff and $\mathcal{O}_X$ be a soft sheaf of rings on X. Then any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ on X is soft.*

Proof. Let $Z \subseteq X$ be closed and $s \in \mathcal{F}|_Z(Z)$. We can extend s to some $U \supseteq Z$ open. Because $Z \coprod X \backslash U$ is closed, the section $(0,1) \in \mathcal{O}_X(X \coprod X \backslash U) = \mathcal{O}_X(X) \times \mathcal{O}_X(X \backslash U)$ extends to $f \in \mathcal{O}_X(X)$ because $\mathcal{O}_X$ is soft. Then $f \cdot s$ extends by 0 to a section on $\mathcal{F}(X)$. $\square$

5 de Rham cohomology

Suppose M is a smooth manifold and denote for any open $U \subseteq M$ the $C^\infty(U,\mathbb{R})$-module $\Omega^i_M(U)$ of i-forms on U. We may form a cochain complex of sheaves

$$0 \longrightarrow \Omega^0_M \longrightarrow \Omega^1_M \longrightarrow \cdots \longrightarrow \Omega^n_M \longrightarrow 0$$

called the de Rham complex. The maps are the exterior derivative d and n is the dimension of M. The de Rham cohomology groups are defined as

$$H^i_{\mathrm{dR}}(M) := \frac{\ker d^i}{\mathrm{im}\ d^{i-1}}$$

which we can now compare against sheaf cohomology.

Theorem. *If M is a smooth manifold, then $H^i_{\mathrm{dR}}(M) \cong H^i(M, \underline{\mathbb{R}})$.*

Proof. The de Rham complex is is exact everywhere, *except* at $i = 0$, by the Poincare Lemma. The kernel of the map $\Omega^0_M \to \Omega^1_M$ is the space of locally constant functions on M, but this is just $\underline{\mathbb{R}}$. Thus we have a resolution of $\underline{\mathbb{R}}$

$$0 \longrightarrow \underline{\mathbb{R}} \longrightarrow \Omega^0_M \longrightarrow \Omega^1_M \longrightarrow \cdots \longrightarrow \Omega^n_M \longrightarrow 0.$$

Using partitions of unity, one can show that any section $s \in C^\infty|_Z(-,\mathbb{R})(Z)$ of a closed $Z \subseteq M$ can be extended to M. Thus $C^\infty(-,\mathbb{R})$ is soft. But Ω^i_M are sheaves of $C^\infty(-,\mathbb{R})$ modules, meaning they are soft too, and hence, $H^0(M,-)$-acyclic. Thus

$$\begin{aligned} H^i(M,\underline{\mathbb{R}}) &:= R^i\Gamma(M,\underline{\mathbb{R}}) \\ &\cong H^i(\Gamma(M,\Omega^\bullet_M)) \\ &\cong H^i(\Omega^\bullet_M(M)) \\ &=: H^i_{dR}(M), \end{aligned}$$

which concludes the proof. $\square$