Seminar Topos Theory
Lecture 10: Čech cohomology

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In this week's lecture, we will discuss Čech cohomology, a tool for computing sheaf cohomology via covering families. For a covering family $\mathcal{U}$ of an object X of a site, we will define a simplicial presheaf $\check{N}\mathcal{U}$, called the *Čech nerve* of $\mathcal{U}$, which should be a simplicial (and thus 'combinatorial') approximation of X. We define Čech cohomology of $\mathcal{U}$ as the simplicial cohomology of the Čech nerve $\check{N}\mathcal{U}$. If the cover $\mathcal{U}$ is in some sense well-behaved, this will compute the sheaf cohomology of X. More generally, we can take the colimit over all covers $\mathcal{U}$, and this will compute sheaf cohomology in some more cases, such as for paracompact Hausdorff spaces or for quasi-coherent sheaves with respect to the Zariski topology on a Noetherian separated scheme.

Throughout, we let $(\mathscr{C}, J)$ be a site and we assume that $\mathscr{C}$ has pullbacks. Then the slice category $\mathscr{C}_{/X}$ over any object $X \in \mathscr{C}$ has products, which are given as pullbacks over X in $\mathscr{C}$. In particular, $\mathscr{C}$ may be the poset of open subsets of a topological space X with its usual Grothendieck topology, where pullbacks are intersections. We write $\mathscr{E} = \mathbf{Sh}(\mathscr{C}, J)$ and $\mathscr{P} = \mathbf{PSh}(\mathscr{C})$ for the corresponding categories of sheaves and presheaves, and denote the canonical embedding $\mathscr{E} \hookrightarrow \mathscr{P}$ by i.

Recall from Lecture 8:

- Sheaf cohomology $H^n(\mathscr{E}, -)$ of $\mathscr{E}$ is the nth right derived functor of the global sections functor $\Gamma = \mathrm{Hom}_{\mathbf{Ab}(\mathscr{E})}(\underline{\mathbb{Z}}, -)$.
- For any sheaf of abelian groups $\mathscr{F} \in \mathbf{Ab}(\mathscr{E})$, we write $H^n(\mathscr{F}, -)$ for the nth right derived functor of $\mathrm{Hom}_{\mathbf{Ab}(\mathscr{E})}(\mathscr{F}, -)$. In particular, for an object X of $\mathscr{C}$, we write $H^n(X, -)$ for $H^n(\mathbb{Z}[i^* \mathrm{y}_X], -)$; by the Yoneda lemma, this is the nth right derived functor of the evaluation functor $\Gamma(X, -) : \mathscr{G} \mapsto \mathscr{G}(X)$.

The exposition here is mostly based on [Joh77, § 8.2].

Notation For a family of maps $\{U_i \to X\}_{i \in I}$ in $\mathscr{C}$ and a sequence $\underline{i} = (i_0, \dots, i_n)$ of elements of I, we write

$$U_{\underline{i}} = U_{i_0, \dots, i_n} := U_{i_0} \underset{X}{\times} \cdots \underset{X}{\times} U_{i_n}$$

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for the pullback of the maps $U_{i_k} \to U$. For topological spaces, this is the union

$$U_{\underline{i}} = U_{i_0,\dots,i_n} = U_{i_0} \cap \cdots \cap U_{i_n}.$$

Note that pullbacks are commutative (up to isomorphism), so $U_{\underline{i}} \cong U_{\sigma\cdot\underline{i}}$ for any permutation $\sigma \in \Sigma_{n+1}$.

1 Čech cohomology of families

We first describe some general constructions we will need. Some of these appear, sometimes in more generality, in the 'Simplicial cheat-sheet'[1].

Construction 1.1 Let $\iota : \mathbf{\Delta} \hookrightarrow \mathbf{Set}$ denote the non-full inclusion; this is a cosimplicial object in $\mathbf{Set}$. It induces a functor $\iota^* : \mathbf{Set} \to \mathbf{sSet}$, sending a set S to the simplicial set $S^\times := \iota^* S$ defined by $[n] \mapsto \mathrm{Hom}_{\mathbf{Set}}([n], S)$.[2] The n-simplices are thus

$$S^\times_n = \mathrm{Hom}_{\mathbf{Set}}([n], S) \cong S^{n+1},$$

and under this isomorphism, the face and degeneracy maps are given by

$$d_k(x_0, \dots, x_n) = (x_0, \dots, \hat{x}_k, \dots, x_n) \quad \text{and} \quad s_k(x_0, \dots, x_n) = (x_0, \dots, x_k, x_k, \dots, x_n).$$

Construction 1.2 Let $\mathcal{U} = \{U_i \to X\}_{i\in I}$ be a family of maps in $\mathcal{C}$ with codomain X. We define the *Čech nerve* $\check{N}\mathcal{U}$ of $\mathcal{U}$ to be the simplicial object in $\mathcal{P}$ with n-simplices

$$(\check{N}\mathcal{U})_n := \coprod_{\underline{i}\in I^\times_n} \mathsf{y}_{U_{\underline{i}}}$$

and with structure maps induced by those of $I^\times$ (the simplicial set of Construction 1.1).

Construction 1.3 Let X be a simplicial object in an abelian category $\mathcal{A}$. We define its *alternating face map complex* $C_\bullet X$ as the chain complex with $C_n X := X_n$ and with differential $\partial_n := \sum_{k=0}^n (-1)^k d_k$. It follows from the simplicial identities that this is a chain complex, i.e., that $\partial^2 = 0$. This defines a functor $C_\bullet : \mathbf{s}\mathcal{A} \to \mathbf{Ch}_{\geqslant 0}(\mathcal{A})$.[3]

Definition 1.4 Let $\mathcal{U} = \{U_i \to X\}_{i\in I}$ be a family of maps in $\mathcal{C}$ with codomain X. Define the *nerve complex* $N_\bullet(\mathcal{U})$ in $\mathbf{Ab}(\mathcal{P})$ as

$$N_\bullet(\mathcal{U}) := C_\bullet(\mathbb{Z}[\check{N}\mathcal{U}]),$$

the alternating face map complex of free abelian group object of the nerve of $\mathcal{U}$. It is the image of the nerve $\check{N}\mathcal{U}$ under the composite

$$\mathbf{s}\mathcal{P} \xrightarrow{\mathbb{Z}[-]} \mathbf{sAb}(\mathcal{P}) \xrightarrow{C_\bullet} \mathbf{Ch}_{\geqslant 0}(\mathbf{Ab}(\mathcal{P})).$$

The nth term in the nerve complex is

$$N_n(\mathcal{U}) = \bigoplus_{\underline{i}\in I^{n+1}} \mathbb{Z}[\mathsf{y}_{U_{\underline{i}}}].$$

[1] Available at https://leoguetta.github.io/simp.pdf.

[2] This is an application of 1.8 of the 'Simplicial cheat-sheet'. The notation $S^\times$ is nonstandard.

[3] This is 2.1 of the 'Simplicial cheat-sheet'. For a topological space, the alternating face map complex of the singular simplicial set $\mathrm{Sing}\, X$ (see Example 1.10 of the 'Simplicial cheat-sheet') is the singular complex of X, and its homology and cohomology are singular homology and cohomology.

Lemma 1.5 *The sequence* $\ldots \to N_1(\mathcal{U}) \to N_0(\mathcal{U})$ *is exact in* $\mathbf{Ab}(\mathcal{P})$.

Proof Exactness in $\mathbf{Ab}(\mathcal{P})$ is pointwise, so it suffices to prove that $N_\bullet(\mathcal{U})(V)$ is exact in $\mathbf{Ab}$ for all $V \in \mathcal{C}$. The nth term $N_n(\mathcal{U})(V)$ is the free abelian group generated by $\coprod_{\underline{i}\in I^{n+1}} \mathcal{C}(V, U_{\underline{i}})$. We have isomorphisms

$$\begin{aligned}
\coprod_{\underline{i}\in I^{n+1}} \mathcal{C}(V, U_{\underline{i}}) &\cong \coprod_{\underline{i}\in I^{n+1}} \mathcal{C}(V, U_{i_0}) \underset{\mathcal{C}(V,X)}{\times} \cdots \underset{\mathcal{C}(V,X)}{\times} \mathcal{C}(V, U_{i_n}) \\
&\cong \coprod_{f\in\mathcal{C}(V,X)} \coprod_{\underline{i}\in I^{n+1}} \mathcal{C}_{/X}(V, U_{i_0}) \times \cdots \times \mathcal{C}_{/X}(V, U_{i_n}) \\
&\cong \coprod_{f\in\mathcal{C}(V,X)} \underbrace{\Big(\coprod_{i\in I} \mathcal{C}_{/X}(V, U_i)\Big)}_{S_f :=}{}^{n+1} \\
&\cong \coprod_{f\in\mathcal{C}(V,X)} S_f^{n+1}.
\end{aligned}$$

It now suffices to prove that a sequence of the form

$$\ldots \to \mathbb{Z}[S^3] \to \mathbb{Z}[S^2] \to \mathbb{Z}[S]$$

is exact; this is the alternating face map complex associated to the simplicial set $S^\times$. This is clear if S is empty; otherwise, we can choose an element $t \in S$ and define a contracting homotopy by the maps

$$\mathbb{Z}[S^n] \to \mathbb{Z}[S^{n+1}], \quad (s_1, \ldots, s_n) \mapsto (t, s_1, \ldots, s_n). \qquad \square$$

Construction 1.6 Let $C_\bullet$ be a chain complex in an abelian category $\mathcal{A}$ and let A be an object of $\mathcal{A}$. Applying the (additive) hom-functor $\mathrm{Hom}_{\mathcal{A}}(-, A)$ levelwise, we obtain a cochain complex $C_A^\bullet$ of abelian groups with

$$C_A^n := \mathrm{Hom}_{\mathcal{A}}(C_n, A).$$

Definition 1.7 The *Čech complex* $\check{C}^\bullet(\mathcal{U}, F)$ of a presheaf of abelian groups F with respect to a family of maps $\mathcal{U} = \{U_i \to X\}_{i\in I}$ is defined as

$$\check{C}^\bullet(\mathcal{U}, F) := \mathrm{Hom}_{\mathbf{Ab}(\mathcal{P})}(N_\bullet(\mathcal{U}), F),$$

a cochain complex in $\mathbf{Ab}$ obtained by applying Construction 1.6 to $N_\bullet(\mathcal{U})$. The nth term of the Čech complex is

$$\check{C}^n(\mathcal{U}, F) = \mathrm{Hom}_{\mathbf{Ab}(\mathcal{P})}\Big(\bigoplus_{\underline{i}\in I^{n+1}} \mathbb{Z}[y_{U_{\underline{i}}}], F\Big) \cong \prod_{\underline{i}\in I^{n+1}} F(U_{\underline{i}}).$$

We write $\check{\partial}$ for the differential in the Čech complex. The nth cohomology of the Čech complex $\check{C}^\bullet(\mathcal{U}, F)$ is the *nth Čech cohomology* of F with respect to $\mathcal{U}$ and denoted $\check{H}^n(\mathcal{U}, F)$.

Remark 1.8 In the case of (pre)sheaves on topological spaces, there is a more efficient description of the Čech complex: instead of indexing it by all $(n + 1)$-tuples of the indexing set I, we can put a partial order on I and index by increasing sequences $i_0 < i_1 < \cdots < i_n$ of elements in I. We write $\check{C}^\bullet_{\mathrm{ord}}(\mathcal{U}, F)$ for this *ordered* variant. If I is finite, then the ordered Čech complex will vanish in degrees $n \geqslant \#I$, and so does the cohomology. One can show that the ordered Čech complex is chain homotopy equivalent to the 'full' Čech complex, so they compute the same cohomology [Con]. This works for the site of open subsets of a topological space, but fails for general Grothendieck topologies.[4]

Remark 1.9 In degree zero, we have

$$\begin{aligned}\check{H}^0(\mathcal{U}, F) &= \ker \check{\partial}_0 = \ker(d_0 - d_1)\\ &= \mathrm{eq}\Big(\prod_{i\in I} F(U_i) \rightrightarrows \prod_{i,j\in I} F(U_{i,j})\Big)\\ &\cong \mathrm{Hom}_{\mathcal{P}}(\widehat{\mathcal{U}}, F),\end{aligned}$$

where $\widehat{\mathcal{U}}$ is the sieve generated by $\mathcal{U}$. Thus, if F satisfies the sheaf axiom for $\mathcal{U}$, we have

$$\check{H}^0(\mathcal{U}, F) \cong \mathrm{Hom}_{\mathcal{P}}(\widehat{\mathcal{U}}, F) \cong \mathrm{Hom}_{\mathcal{P}}(\mathsf{y}_X, F) \cong F(X).$$

Here the second isomorphism is the sheaf axiom and the third is the Yoneda lemma.

Proposition 1.10 *The nth Čech cohomology $\check{H}^n(\mathcal{U}, -)$ with respect to $\mathcal{U}$ is the nth right derived functor of $\check{H}^0(\mathcal{U}, -) : \mathbf{Ab}(\mathcal{P}) \to \mathbf{Ab}$.*

Remark 1.11 This statement is not true for the category of sheaves $\mathcal{E}$: the right derived functors of the composite $\check{H}^0(\mathcal{U}, i_*(-)) : \mathbf{Ab}(\mathcal{E}) \to \mathbf{Ab}$ do not coincide in general with the composite $\check{H}^n(\mathcal{U}, i_*(-))$. The obstruction is that i_* might not be exact. In fact, the composite $\check{H}^0(\mathcal{U}, i_*(-))$ is the functor $\Gamma(X, -) := \mathrm{Hom}_{\mathcal{P}}(\mathsf{y}_X, i_*(-))$ (cf. Remark 1.9), whose right derived functors are ordinary sheaf cohomology.

Proof (of Proposition 1.10) The complex $N_\bullet(\mathcal{U})$ consists of coproducts of representables, which are projective as we saw last week. Since the hom-functor $\mathrm{Hom}_{\mathbf{Ab}(\mathcal{P})}(P, -)$ is exact if P is projective, the functor $F \mapsto \check{C}^\bullet(\mathcal{U}, F) = \mathrm{Hom}_{\mathbf{Ab}(\mathcal{P})}(N_\bullet(\mathcal{U}), F)$, assigning to an abelian presheaf the associated Čech complex, is exact. Hence, the functors $\check{H}^n(\mathcal{U}, -)$ form a δ-functor. Moreover, if I is an injective presheaf, then $\mathrm{Hom}_{\mathbf{Ab}(\mathcal{P})}(-, I)$ is exact, so the Čech complex $\check{C}^\bullet(\mathcal{U}, I)$ is exact in positive degrees by Lemma 1.5, and thus the cohomology $\check{H}^n(\mathcal{U}, I)$ vanishes for $n > 0$. The result now follows from the universal property of derived functors, stated last week. □

The following result provides a criterion for a covering family $\mathcal{U}$ on an object X making Čech cohomology of $\mathcal{U}$ compute sheaf cohomology of X.

Proposition 1.12 *Let $\mathcal{U} = \{U_i \hookrightarrow X\}_{i\in I}$ be covering family of an object X in a site $(\mathcal{C}, J)$ and let $\mathcal{F}$ be a sheaf of abelian groups on $(\mathcal{C}, J)$. If $H^n(U_{\underline{i}}, \mathcal{F}) = 0$ for all $n > 0$ and $\underline{i} \in I^{m+1}$, then there is an isomorphism*

$$\check{H}^n(\mathcal{U}, \mathcal{F}) \cong H^n(X, \mathcal{F})$$

between Čech cohomology and sheaf cohomology for all $n \geqslant 0$.

[4] See https://mathoverflow.net/questions/10056/equivalence-of-ordered-and-unordered-cech-cohomology.

The first homework exercise is to prove this result.

2 Čech cohomology of sites

We now discuss the functoriality of $\check{H}^n(\mathcal{U}, F)$ in the family $\mathcal{U}$. We will show that the groups $\check{H}^n(-, F)$ form a diagram over the poset $J(X)$ of covering sieves of X, and we will define the Čech cohomology of the object X as the colimit of this diagram. The idea is thus to look at finer and finer covers of X; it turns out that this computes ordinary sheaf cohomology in low degrees, and in all degrees under some more assumptions.

Definition 2.1 Let $\mathcal{U} = \{U_i \to X\}_{i\in I}$ and $\mathcal{V} = \{V_j \to X\}_{j\in J}$ be two families of maps in $\mathcal{C}$ with codomain X. A *refinement map* $r : \mathcal{V} \to \mathcal{U}$ is a function $r : J \to I$ with factorisations

$$\begin{tikzcd} V_j \arrow[rr, "r_j"] \arrow[dr] & & U_{r(j)} \arrow[dl] \\ & X & \end{tikzcd}$$

for all $j \in J$.

For $\underline{j} \in J^{n+1}$, a refinement map $r : \mathcal{V} \to \mathcal{U}$ induces a map $r_{\underline{j}} : V_{\underline{j}} \to U_{r(\underline{j})}$ over X; thus r induces a chain map $N_\bullet(r) : N_\bullet(\mathcal{V}) \to N_\bullet(\mathcal{U})$.

The following lemma tells us that, up to chain homotopy, all refining maps induce the same chain map.

Lemma 2.2 *Let $\mathcal{U} = \{U_i \to X\}_{i\in i}$ and $\mathcal{V} = \{V_j \to X\}_{j\in J}$ be two families of maps in $\mathcal{C}$ with codomain X, and let $r, s : \mathcal{V} \to \mathcal{U}$ be two refinement maps. Then the chain maps $N_\bullet(r)$ and $N_\bullet(s)$ are chain homotopic.*

Proof For $\underline{j} \in J^{n+1}$ and $k \in [n]$, define a map $t^k_{\underline{j}}$ over X as

$$t^k_{\underline{j}} := (r_{j_0}, \dots, r_{j_k}, s_{j_k}, \dots, s_{j_n}) : V_{\underline{j}} \to U_{(r(j_0),\dots,r(j_k),s(j_k),\dots,s(j_n))}.$$

Let $t^k_n : N_n(\mathcal{V}) \to N_{n+1}(\mathcal{U})$ be the direct sum over $\underline{j} \in J^{n+1}$ of the maps induced by $t^k_{\underline{j}}$. One can now check that the alternating sums $\sum_{k=0}^n (-1)^k t^k_n$ form a chain homotopy from $N_\bullet(r)$ to $N_\bullet(s)$ (and this is tedious combinatorics). □

Corollary 2.3 *Let $\mathcal{U} = \{U_i \to X\}_{i\in i}$ be a family of maps in $\mathcal{C}$ with codomain X and let $\widehat{\mathcal{U}} \subseteq \mathrm{y}_X$ denote the sieve generated by $\mathcal{U}$. Then there is a canonical isomorphism $\check{H}^n(\mathcal{U}, F) \cong \check{H}^n(\widehat{\mathcal{U}}, F)$ for every presheaf of abelian groups F.*

Proof By definition, every morphism in $\widehat{\mathcal{U}}$ factors through one of the maps $U_i \to X$, so there is a refinement map $\widehat{\mathcal{U}} \to \mathcal{U}$. On the other hand, the inclusion map defines a refinement map $\mathcal{U} \to \widehat{\mathcal{U}}$. By Lemma 2.2, there is a chain homotopy equivalence between $N_\bullet(\mathcal{U})$ and $N_\bullet(\widehat{\mathcal{U}})$, and hence between $\check{C}^\bullet(\mathcal{U}, F)$ and $\check{C}^\bullet(\widehat{\mathcal{U}}, F)$, and thus their cohomology groups $\check{H}^n(\mathcal{U}, F)$ and $\check{H}^n(\widehat{\mathcal{U}}, F)$ are isomorphic. Moreover, any room for choice of refinement map $\widehat{\mathcal{U}} \to \mathcal{U}$ gets killed in the process by the fact that the resulting cochain maps are homotopic: by Exercise 1 of Homework 6, they then induce the same map on cohomology. □

We will now restrict to the case where $\mathcal{U}$ is a J-covering sieve. By Corollary 2.3, however, we may replace a covering sieve by a family generating it to compute Čech cohomology, which could be easier (the second homework exercise should convince you of this). If R and S are any sieves on an object X of $\mathscr{C}$, there exists a refinement map $R \to S$ if and only if $R \subseteq S$. Therefore, the $N_\bullet(\mathcal{U})$ form a diagram over $\mathcal{U} \in J(X)$, seen as a poset ordered by inclusion; thus, $\check{C}^\bullet(\mathcal{U}, A)$ and $\check{H}^n(\mathcal{U}, A)$ form diagrams over $\mathcal{U} \in J(X)^{\mathrm{op}}$.

Definition 2.4 Let X be an object of $\mathscr{C}$ and let F be presheaf of abelian groups on $\mathscr{C}$. Then the *nth Čech cohomology* $\check{H}^n(X, F)$ of X with values in F is the (filtered) colimit

$$\check{H}^n(X, F) := \operatorname*{colim}_{\mathcal{U} \in J(X)^{\mathrm{op}}} \check{H}^n(\mathcal{U}, F).$$

Remark 2.5 Since $\check{H}^0(\mathcal{U}, F) = \mathrm{Hom}_{\mathscr{P}}(\mathcal{U}, F)$ by Remark 1.9, it follows from the definition that $\check{H}^0(X, -)$ recovers the 'half-sheafification' functor $(-)^+$ from Lecture 2.

Remark 2.6 Since filtered colimits are exact, the functors $\check{H}^n(X, -) : \mathbf{Ab}(\mathscr{P}) \to \mathbf{Ab}$ form a δ-functor, and they vanish on injectives in positive degrees since the $\check{H}^n(\mathcal{U}, -)$ do (as we saw in the proof of Proposition 1.10). Thus, by the universal property of right derived functors, we have

$$\check{H}^n(X, -) \cong \mathrm{R}^n\check{H}^0(X, -).$$

We are mostly interested in Čech cohomology of sheaves; however, the functors $\check{H}^n(X, -)$ do not form a δ-functor $\mathbf{Ab}(\mathscr{E}) \to \mathbf{Ab}$ since i_* might not be exact (see Remark 1.11).

3 Čech cohomology and sheaf cohomology

We have now defined a notion of cohomology, which we can apply to sheaves. One might naturally wonder how this notion compares to the notion of sheaf cohomology we defined in earlier lectures: although the constructions involved are maybe not immediately completely transparent, we can compute Čech cohomology as the cohomology of some explicitly defined complex. We will see that Čech cohomology always agrees with sheaf cohomology in degrees zero and one. In higher degrees this fails in general (even for sheaves on topological spaces). There are, however, some assumptions that are satisfied by many examples of interest under which Čech cohomology and sheaf cohomology will agree in all degrees; for instance, we will prove that they agree for sheaves on paracompact Hausdorff topological spaces.

The following theorem relates Čech cohomology and sheaf cohomology without any additional assumptions on the site or sheaf: they agree in degrees zero and one, but might not in higher degrees.

Theorem 3.1 ([Joh77, Theorem 8.27]) *Let X be an object of $\mathscr{C}$ and let $\mathscr{F}$ be a sheaf of abelian groups on a site $(\mathscr{C}, J)$. Then there is a map*

$$\check{H}^n(X, \mathscr{F}) \to H^n(X, \mathscr{F})$$

which is an isomorphism for $n = 0, 1$ and a monomorphism for $n = 2$.

The proof uses some more technology than we have introduced: either the Grothendieck spectral sequence (a special case of which we will use, cf. Proposition 4.2) or a close inspection of a double complex.

We state without proof the following result, due to Cartan, providing a criterion for Čech cohomology and sheaf cohomology to agree in all degrees.

Proposition 3.2 ([Joh77, Proposition 8.28]) *Let $\mathcal{F}$ be a sheaf of abelian groups on $(\mathcal{C}, J)$ and suppose there is a set K of objects of $\mathcal{C}$ with the following properties:*

① *for every $V \in K$, we have $\check{H}^n(V, \mathcal{F}) = 0$ for $n > 0$;*

② *for every $U \in \mathcal{C}$, there is a covering family $\{V_i \to U\}_{i \in I}$ of U with $V_i \in K$ for all $i \in I$;*

③ *for all $V, W \in K$, the pullback $V \times_U W$ is an element of K.*

Then the map $\check{H}^n(X, \mathcal{F}) \to H^n(X, \mathcal{F})$ of Theorem 3.1 is an isomorphism for all $n \geqslant 0$ and all $X \in \mathcal{C}$.

4 Čech cohomology and sheaf cohomology for paracompact Hausdorff spaces

The next theorem relates Čech cohomology and sheaf cohomology in all degrees, but only for sheaves on topological spaces that are assumed to be paracompact Hausdorff.

Theorem 4.1 *Let X be a paracompact Hausdorff topological space. Then for every sheaf of abelian groups $\mathcal{F}$ on X there is an isomorphism*

$$\check{H}^n(X, \mathcal{F}) \cong H^n(X, \mathcal{F})$$

between Čech cohomology and sheaf cohomology in all degrees $n \geqslant 0$.

The proof will depend crucially on an edge case of the *Grothendieck spectral sequence*, relating $\mathrm{R}^n G \circ \mathrm{R}^m F$ to $\mathrm{R}^{m+n}(GF)$. This can be proven 'by hand' using techniques that have been discussed before (splicing up an injective resolution, looking at the correct long exact sequences).

Proposition 4.2 *Let $F : \mathcal{A} \to \mathcal{B}$ and $G : \mathcal{B} \to \mathcal{C}$ be left exact functors between abelian categories, where $\mathcal{A}$ and $\mathcal{B}$ are assumed to have enough injectives and F sends injective objects to G-acyclic objects.*

① *If $X \in \mathcal{A}$ satisfies $\mathrm{R}^n G(\mathrm{R}^m F(X)) = 0$ for all $m \geqslant 0$ and $n > 0$, then*

$$\mathrm{R}^n(GF)(X) \cong G(\mathrm{R}^n F(X))$$

for all $n \geqslant 0$.

② *If $X \in \mathcal{A}$ satisfies $\mathrm{R}^n G(\mathrm{R}^m F(X)) = 0$ for all $m > 0$ and $n \geqslant 0$, then*

$$\mathrm{R}^n(GF)(X) \cong \mathrm{R}^n G(F(X))$$

for all $n \geqslant 0$.

Lemma 4.3 *If F is a presheaf of abelian groups on a paracompact Hausdorff space X with vanishing sheafification $i^*F = 0$,*[5] *then $\check{H}^n(X, F) = 0$ for all $n \geqslant 0$.*

The proof is point-set topology: here you have to use the assumptions on the topological space. An outline of the proof can be found in [Joh77, Exercise 8.9].

Lemma 4.4 *Let $\mathbb{H}^n$ denote the nth right derived functor of $i_* : \mathbf{Ab}(\mathscr{E}) \to \mathbf{Ab}(\mathscr{P})$. Then $i^*\mathbb{H}^n(\mathscr{F}) = 0$ for all $n > 0$ and $\mathscr{F} \in \mathbf{Ab}(\mathscr{E})$.*

Proof Consider the functors

$$\mathbf{Ab}(\mathscr{E}) \xrightarrow{i_*} \mathbf{Ab}(\mathscr{P}) \xrightarrow{i^*} \mathbf{Ab}(\mathscr{E}).$$

The former is left exact and the latter is exact; their composite is (equivalent to) the identity, which is exact. It follows that these functors satisfy the hypotheses of part ① of Proposition 4.2, and we find

$$i^*\mathbb{H}^n(\mathscr{F}) = i^*(\mathrm{R}^n i_*(\mathscr{F})) \cong \mathrm{R}^n(i^*i_*)(\mathscr{F}) = \mathrm{R}^n\mathrm{id}(\mathscr{F}),$$

which vanishes for $n > 0$ by exactness of the identity. □

Proof (of Theorem 4.1) Consider the left exact functors

$$\mathbf{Ab}(\mathbf{Sh}(X)) \xrightarrow{i_*} \mathbf{Ab}(\mathbf{PSh}(X)) \xrightarrow{\check{H}^0(X,-)} \mathbf{Ab}.$$

Their composite is the global sections functor $H^0(X, -) = \Gamma(X, -)$ (this follows from Remark 2.5). We have $i^*\mathbb{H}^m(\mathscr{F}) = 0$ for all $m > 0$ by Lemma 4.4, so we see

$$\check{H}^n(X, \mathbb{H}^m(\mathscr{F})) = 0$$

for all $m > 0$ and $n \geqslant 0$ by Lemma 4.3. By part ② of Proposition 4.2, we conclude

$$H^n(X, \mathscr{F}) = \mathrm{R}^n H^0(X, -)(\mathscr{F}) = \mathrm{R}^n(\check{H}^0(X, -) \circ i_*)(\mathscr{F}) \cong \mathrm{R}^n\check{H}^0(X, -)(\mathscr{F}) \cong \check{H}^n(X, \mathscr{F}),$$

using the fact that $\check{H}^n(X, -)$ is the nth right derived functor of $\check{H}^0(X, -)$ (Remark 2.6) and that i_* preserves injectives (which is part of the homework). □

[5] Equivalently, we may require that all stalks of F vanish.

Homework exercises

Exercise 10.1 Let $\mathcal{U} = \{U_i \hookrightarrow X\}_{i\in I}$ be a covering family of an object X in a site $(\mathscr{C}, J)$ and let $\mathcal{F}$ be a sheaf of abelian groups on $(\mathscr{C}, J)$. Assume that $H^n(U_{\underline{i}}, \mathcal{F}) = 0$ for all $n > 0$ and $\underline{i} \in I^{m+1}$. As before, write $\mathscr{E} = \mathbf{Sh}(\mathscr{C}, J)$ and $\mathscr{P} = \mathbf{PSh}(\mathscr{C})$.

① Show the following statement: if $\mathscr{A}$ and $\mathscr{B}$ are abelian categories and if

$$\begin{tikzcd} \mathscr{A} \arrow[r, shift left, "F"] \arrow[r, phantom, "\perp"] & \mathscr{B} \arrow[l, shift left, "U"] \end{tikzcd}$$

is an adjunction where F is exact, then U preserves injective objects. Deduce that $i_* : \mathbf{Ab}(\mathscr{E}) \to \mathbf{Ab}(\mathscr{P})$ preserves injective objects.

② Let $\mathbb{H}^m : \mathbf{Ab}(\mathscr{E}) \to \mathbf{Ab}(\mathscr{P})$ denote the mth right derived functor of i_*. Show that for any sheaf $\mathscr{G} \in \mathbf{Ab}(\mathscr{E})$, the presheaf $\mathbb{H}^m(\mathscr{G})$ is given by $U \mapsto H^m(U, \mathscr{G})$.

Hint: Use Proposition 4.2.

③ Show that $\check{H}^n(\mathcal{U}, \mathbb{H}^m(\mathcal{F})) = 0$ for all $m > 0$ and $n \geqslant 0$.

④ Conclude that there is an isomorphism $\check{H}^n(\mathcal{U}, \mathcal{F}) \cong H^n(X, \mathcal{F})$ between Čech cohomology and sheaf cohomology in all degrees $n \geqslant 0$.

Hint: Use Remark 1.9 and Proposition 4.2.

Exercise 10.2 Use Čech cohomology to compute the sheaf cohomology $H^n(S^1, \underline{\mathbb{Z}})$ of the circle S^1 with coefficients in the constant sheaf $\underline{\mathbb{Z}}$ for all $n \geqslant 0$. You may use without proof the following fact, which is a weak version of homotopy invariance of sheaf cohomology: if an open subset $U \subseteq S^1$ is a disjoint union of contractibles, then $H^n(U, \underline{\mathbb{Z}}) = 0$ for all $n > 0$.

Hint: Use the first exercise, Remark 1.8, Corollary 2.3, and the fact that $\underline{\mathbb{Z}}(U) \cong \mathrm{Hom}_{\mathbf{Top}}(U, \mathbb{Z})$ is the set of continuous maps $U \to \mathbb{Z}$, where the codomain is equipped with the discrete topology.

References

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