Seminar Topos Theory: Sheaf Cohomology (WISM567)

Lecture 11: Hypercoverings

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In this week's lecture, we will discuss hypercoverings in sites, which are a generalization of coverings. Recall last week we proved that if X is a paracompact Hausdorff topological space, then for every sheaf of abelian groups $\mathscr{F}$ on X there is an isomorphism $\check{H}^n(X, \mathscr{F}) \cong H^n(X, \mathscr{F})$ between Čech cohomology and sheaf cohomology in all degrees $n \geqslant 0$. But in general, if X is an object of a site $\mathcal{C}$, Čech cohomology groups don't coincide with sheaf cohomology groups. However, one can refine the notion of coverings to that of hypercoverings and compute the hyper Čech cohomology, which is related to the sheaf cohomology, by the Verdier hypercovering theorem. We mainly follow the Stacks project chapter 25 ([Sta25, Tag 01FX]) and the ultimate goal of this lecture is to prove the Verdier hypercovering theorem. Due to time constraints, we will not cover the hypercoverings of spaces and the construction of hypercoverings, which is the last two sections of the Stacks project chapter 25 ([Sta25, Tag 01H1] and [Sta25, Tag 094J]). Other classical references are [AGV72, Exposé V, §7] and [DHI04].

I Semi-representable objects

Definition I.1. *Let $\mathcal{C}$ be a category. We denote $\mathrm{SR}(\mathcal{C})$ the category of semi-representable objects defined as follows:*

(1) objects are families of objects $\{U_i\}_{i \in I}$, and

(2) morphisms $\{U_i\}_{i\in I} \to \{V_j\}_{j \in J}$ are given by a map $\alpha : I \to J$ and for each $i \in I$ a morphism $f_i : U_i \to V_{\alpha(i)}$ of $\mathcal{C}$.

Let $X \in \mathrm{Ob}(\mathcal{C})$. The category of semi-representable objects over X is the category $\mathrm{SR}(\mathcal{C}, X) = \mathrm{SR}(\mathcal{C}/X)$.

Note that the objects and morphisms in $\mathrm{SR}(\mathcal{C}, X)$ are:

(1) objects are families of morphisms $\{U_i \to X\}_{i\in I}$, and

(2) morphisms $\{U_i \to X\}_{i\in I} \to \{V_j \to X\}_{j\in J}$ are given by a map $\alpha : I \to J$ and for each $i \in I$ a morphism $f_i : U_i \to V_{\alpha(i)}$ over X.

There is a forgetful functor $\mathrm{SR}(\mathcal{C}, X) \to \mathrm{SR}(\mathcal{C})$.

Definition I.2. *Let $\mathcal{C}$ be a category. We denote F the functor which associates a presheaf to a semi-representable object. In a formula*

$$\begin{array}{rcl} F : \mathrm{SR}(\mathcal{C}) & \longrightarrow & \mathrm{PSh}(\mathcal{C}) \\ \{U_i\}_{i\in I} & \longmapsto & \amalg_{i\in I} \mathrm{y}_{U_i} \end{array} \tag{1}$$

where y_U denotes the representable presheaf associated to the object U.

Given a morphism $U \to X$ we obtain a morphism $h_U \to h_X$ of representable presheaves. Thus we often think of F on $\mathrm{SR}(\mathcal{C}, X)$ as a functor into the category of presheaves of sets over h_X, namely $\mathrm{PSh}(\mathcal{C})/h_X$.

Next we discuss the existence of limits in the category of semi-representable objects.

Lemma I.3. *Let $\mathcal{C}$ be a category.*

(1) the category $\mathrm{SR}(\mathcal{C})$ has coproducts and F commutes with them,

(2) the functor $F : \mathrm{SR}(\mathcal{C}) \to \mathrm{PSh}(\mathcal{C})$ commutes with limits,

(3) if $\mathcal{C}$ has fibre products, then $\mathrm{SR}(\mathcal{C})$ has fibre products,

(4) if $\mathcal{C}$ has products of pairs, then $\mathrm{SR}(\mathcal{C})$ has products of pairs,

(5) if $\mathcal{C}$ has equalizers, so does $\mathrm{SR}(\mathcal{C})$, and

(6) if $\mathcal{C}$ has a final object, so does $\mathrm{SR}(\mathcal{C})$.

Let $X \in \mathrm{Ob}(\mathcal{C})$.

(1) the category $\mathrm{SR}(\mathcal{C}, X)$ has coproducts and F commutes with them,

(2) if $\mathcal{C}$ has fibre products, then $\mathrm{SR}(\mathcal{C}, X)$ has finite limits and $F : \mathrm{SR}(\mathcal{C}, X) \to \mathrm{PSh}(\mathcal{C})/h_X$ commutes with them.

Proof. Proof of the statements about $\mathrm{SR}(\mathcal{C})$:

(1) The coproduct of $\{U_i\}_{i\in I}$ and $\{V_j\}_{j\in J}$ is $\{U_i\}_{i\in I} \amalg \{V_j\}_{j\in J}$, in other words, the family of objects whose index set is $I \amalg J$ and for an element $k \in I \amalg J$ gives U_i if $k = i \in I$ and gives V_j if $k = j \in J$. Similarly for coproducts of families of objects. It is clear that F commutes with these.

(2) For $U \in \mathrm{Ob}(\mathcal{C})$ consider the object $\{U\}$ of $\mathrm{SR}(C)$. Consider a diagram $D : J \to \mathrm{SR}(\mathcal{C})$ with a limit $\lim D = \{W_k\}_{k\in K}$ in $\mathrm{SR}(\mathcal{C})$, then

$$F(\lim D)(U) = (\coprod_{k\in K} \mathrm{y}_{W_k})(U) = \coprod_{k\in K} \mathrm{Hom}_{\mathcal{C}}(U, W_k) = \mathrm{Mor}_{\mathrm{SR}(\mathcal{C})}(\{U\}, \lim D)$$

and $(\lim(F \circ D))(U) = \lim_J((F \circ D)(j)(U)) = \lim_J(F(D(j))(U))$.

Suppose $D(j) = \{V_{j,m}\}_{m\in J_j}$ for each $j \in J$, then $F(D(j)) = F\left(\{V_{j,m}\}_{m\in J_j}\right) = \coprod_{m\in J_j} \mathrm{y}_{V_{j,m}}$. Thus $F(D(j))(U) = \coprod_{m\in J_j} \mathrm{Hom}_{\mathcal{C}}(U, V_{j,m}) = \mathrm{Mor}_{\mathrm{SR}(\mathcal{C})}(\{U\}, D(j))$.

Since the functor $\mathrm{Mor}_{\mathrm{SR}(\mathcal{C})}(\{U\},-):\mathrm{SR}(\mathcal{C})\to\mathrm{Sets}$ preserves limits, then

$$\lim_J F(D(j))(U)=\lim_J \mathrm{Mor}_{\mathrm{SR}(\mathcal{C})}(\{U\},D(j))=\mathrm{Mor}_{\mathrm{SR}(\mathcal{C})}(\{U\},\lim D).$$

Therefore $F(\lim D)(U)=\mathrm{Mor}_{\mathrm{SR}(\mathcal{C})}(\{U\},\lim D)=(\lim(F\circ D))(U)$. Since limits of presheaves are computed at the level of sections, we conclude $F(\lim D)=\lim(F\circ D)$.

(3) Suppose given a morphism $(\alpha,f_i):\{U_i\}_{i\in I}\to\{V_j\}_{j\in J}$ and a morphism $(\beta,g_k):\{W_k\}_{k\in K}\to\{V_j\}_{j\in J}$. The fibred product of these morphisms is given by

$$\left\{U_i\times_{f_i,V_j,g_k}W_k\right\}_{(i,j,k)\in I\times J\times K}\text{ such that }j=\alpha(i)=\beta(k)$$

The fibre products exist if $\mathcal{C}$ has fibre products.

(4) The product of $\{U_i\}_{i\in I}$ and $\{V_j\}_{j\in J}$ is $\{U_i\times V_j\}_{i\in I,j\in J}$. The products exist if $\mathcal{C}$ has products.

(5) The equalizer of two maps $(\alpha,f_i),(\alpha',f_i'):\{U_i\}_{i\in I}\to\{V_j\}_{j\in J}$ is

$$\left\{\mathrm{Eq}\left(f_i,f_i':U_i\to V_{\alpha(i)}\right)\right\}_{i\in I,\alpha(i)=\alpha'(i)}$$

The equalizers exist if $\mathcal{C}$ has equalizers.

(6) If X is a final object of $\mathcal{C}$, then $\{X\}$ is a final object of $\mathrm{SR}(\mathcal{C})$.

Proof of the statements about $\mathrm{SR}(\mathcal{C},X)$:

These follow from the results above applied to the category $\mathcal{C}/X$ using that $\mathrm{SR}(\mathcal{C}/X)=\mathrm{SR}(\mathcal{C},X)$ and that $\mathrm{PSh}(\mathcal{C}/X)=\mathrm{PSh}(\mathcal{C})/h_X$. However we also argue directly as follows. It is clear that the coproduct of $\{U_i\to X\}_{i\in I}$ and $\{V_j\to X\}_{j\in J}$ is $\{U_i\to X\}_{i\in I}\amalg\{V_j\to X\}_{j\in J}$ and similarly for coproducts of families of families of morphisms with target X. The object $\{X\to X\}$ is a final object of $\mathrm{SR}(\mathcal{C},X)$. Suppose given a morphism $(\alpha,f_i):\{U_i\to X\}_{i\in I}\to\{V_j\to X\}_{j\in J}$ and a morphism $(\beta,g_k):\{W_k\to X\}_{k\in K}\to\{V_j\to X\}_{j\in J}$. The fibred product of these morphisms is given by

$$\left\{U_i\times_{f_i,V_j,g_k}W_k\to X\right\}_{(i,j,k)\in I\times J\times K}\text{ such that }j=\alpha(i)=\beta(k)$$

The fibre products exist by the assumption that $\mathcal{C}$ has fibre products. Thus $\mathrm{SR}(\mathcal{C},X)$ has finite limits. We omit verifying the statements on the functor F in this case. □

II Skeleton and coskeleton functors

Recall the category Δ is the category with

(1) objects $[0],[1],[2],\ldots$ with $[n]=\{0,1,2,\ldots,n\}$ and

(2) a morphism $[n] \to [m]$ is a nondecreasing map $\{0, 1, 2, \dots, n\} \to \{0, 1, 2, \dots, m\}$ between the corresponding sets.

Here nondecreasing for a map $\varphi : [n] \to [m]$ means by definition that $\varphi(i) \geq \varphi(j)$ if $i \geq j$.

Let $\Delta_{\leq n}$ denote the full subcategory of Δ with objects $[0], [1], [2], \dots, [n]$. Let $\mathcal{C}$ be a category.

Definition II.1. *An n-truncated simplicial object of $\mathcal{C}$ is a contravariant functor from $\Delta_{\leq n}$ to $\mathcal{C}$. A morphism of n-truncated simplicial objects is a transformation of functors. We denote the category of n-truncated simplicial objects of $\mathcal{C}$ by the symbol* $\mathrm{Simp}_n(\mathcal{C})$.

Given a simplicial object U of $\mathcal{C}$ the truncation $\mathrm{sk}_n U$ is the restriction of U to the subcategory $\Delta_{\leq n}$. This defines a skeleton functor

$$\mathrm{sk}_n : \mathrm{Simp}(\mathcal{C}) \longrightarrow \mathrm{Simp}_n(\mathcal{C}) \tag{2}$$

from the category of simplicial objects of $\mathcal{C}$ to the category of n-truncated simplicial objects of $\mathcal{C}$.

Let $\mathcal{C}$ be a category. The coskeleton functor (if it exists) is a functor

$$\mathrm{cosk}_n : \mathrm{Simp}_n(\mathcal{C}) \longrightarrow \mathrm{Simp}(\mathcal{C}) \tag{3}$$

which is right adjoint to the skeleton functor. In a formula

$$\mathrm{Mor}_{\mathrm{Simp}(\mathcal{C})}(U, \mathrm{cosk}_n V) = \mathrm{Mor}_{\mathrm{Simp}_n(\mathcal{C})}(\mathrm{sk}_n U, V)$$

We let $(\Delta/[n])_{\leq m}$ denote the full subcategory of $\Delta/[n]$ consisting of objects $[k] \to [n]$ of $\Delta/[n]$ with $k \leq m$. Given a m-truncated simplicial object U of $\mathcal{C}$ we define a functor

$$U(n) : (\Delta/[n])_{\leq m}^{opp} \longrightarrow \mathcal{C} \tag{4}$$

by the rules

$$([k] \to [n]) \longmapsto U_k$$
$$\psi : ([k'] \to [n]) \to ([k] \to [n]) \longmapsto U(\psi) : U_k \to U_{k'}$$

For a given morphism $\varphi : [n] \to [n']$ of Δ we have an associated functor

$$\bar{\varphi} : (\Delta/[n])_{\leq m} \longrightarrow (\Delta/[n'])_{\leq m}$$

which maps $\alpha : [k] \to [n]$ to $\varphi \circ \alpha : [k] \to [n']$. The composition $U(n') \circ \bar{\varphi}$ is equal to the functor $U(n)$.

Lemma II.2. *If the category $\mathcal{C}$ has finite limits, then* cosk_m *functors exist for all m. Moreover, for any m-truncated simplicial object U the simplicial object* $\operatorname{cosk}_m U$ *is described by the formula*

$$(\operatorname{cosk}_m U)_n = \lim_{(\Delta/[n])^{opp}_{\leq m}} U(n) \tag{5}$$

and for $\varphi : [n] \to [n']$ the map $\operatorname{cosk}_m U(\varphi)$ *comes from the identification $U(n') \circ \bar{\varphi} = U(n)$ above.*

Proof. As shown in [Sta25, Tag 0162], Lemma 19.2. $\square$

III Homotopies

For every $n \geq 0$ we denote $\Delta[n]$ the simplicial set

$$\Delta[n] : \Delta^{op} \longrightarrow \text{Sets}, \quad [k] \longmapsto \operatorname{Mor}_\Delta([k], [n]) \tag{6}$$

Consider the simplicial sets $\Delta[0]$ and $\Delta[1]$. Recall that there are two morphisms

$$e_0, e_1 : \Delta[0] \longrightarrow \Delta[1],$$

coming from the morphisms $[0] \to [1]$ mapping 0 to an element of $[1] = \{0, 1\}$. Note that each set $\Delta[1]_k$ is finite. Hence, if the category $\mathcal{C}$ has finite coproducts, then we can form the product

$$U \times \Delta[1]$$

given by $(U \times \Delta[1])_n = \coprod_{\alpha \in \Delta[1]_n} U_n$ for any simplicial object U of $\mathcal{C}$. Note that $\Delta[0]$ has the property that $\Delta[0]_k = \{*\}$ is a singleton for all $k \geq 0$. Hence $U \times \Delta[0] = U$. Thus e_0, e_1 above gives rise to morphisms

$$e_0, e_1 : U \to U \times \Delta[1].$$

Definition III.1. *Let $\mathcal{C}$ be a category having finite coproducts. Suppose that U and V are two simplicial objects of $\mathcal{C}$. Let $a, b : U \to V$ be two morphisms.*

(1) We say a morphism

$$h : U \times \Delta[1] \longrightarrow V$$

is a homotopy from a to b if $a = h \circ e_0$ and $b = h \circ e_1$.

(2) We say the morphisms a and b are homotopic or are in the same homotopy class if there exists a sequence of morphisms $a = a_0, a_1, \ldots, a_n = b$ from U to V such that for each

$i = 1, \ldots, n$ there either exists a homotopy from a_{i-1} to a_i or there exists a homotopy from a_i to a_{i-1}.

The relation "there is a homotopy from a to b " is in general not transitive or symmetric; it is reflexive (See [Sta25, Tag 019J], Example 26.3). While "being homotopic" is an equivalence relation on the set $\mathrm{Mor}(U, V)$ and it is the equivalence relation generated by the relation "there is a homotopy from a to b ". It turns out we can define homotopies between pairs of maps of simplicial objects in any category.

IV Hypercoverings

Definition IV.1. *Let $\mathcal{C}$ be a site. Let $f = (\alpha, f_i) : \{U_i\}_{i \in I} \to \{V_j\}_{j \in J}$ be a morphism in the category $\mathrm{SR}(\mathcal{C})$. We say that f is a covering if for every $j \in J$ the family of morphisms $\{U_i \to V_j\}_{i \in I, \alpha(i) = j}$ is a covering for the site $\mathcal{C}$. Let X be an object of $\mathcal{C}$. A morphism $K \to L$ in $\mathrm{SR}(\mathcal{C}, X)$ is a covering if its image in $\mathrm{SR}(\mathcal{C})$ is a covering.*

Lemma IV.2. *Let $\mathcal{C}$ be a site. (1) A composition of coverings in $\mathrm{SR}(\mathcal{C})$ is a covering.*

(2) If $\mathcal{C}$ has fibre products and $K \to L$ is a covering in $\mathrm{SR}(\mathcal{C})$ and $L' \to L$ is a morphism, then $L' \times {}_L K$ exists and $L' \times_L K \to L'$ is a covering.

(3) If $\mathcal{C}$ has products of pairs, and $A \to B$ and $K \to L$ are coverings in $\mathrm{SR}(\mathcal{C})$, then $A \times K \to B \times L$ is a covering.

Let $X \in \mathrm{Ob}(\mathcal{C})$. Then (1) and (2) holds for $\mathrm{SR}(\mathcal{C}, X)$ and (3) holds if $\mathcal{C}$ has fibre products.

Proof. Homework Exercise VII.1. □

By Lemma I.3 and Lemma II.2, the coskeleton of a truncated simplicial object of $\mathrm{SR}(\mathcal{C}, X)$ exists if $\mathcal{C}$ has fibre products.

Definition IV.3. *Let $\mathcal{C}$ be a site. Assume $\mathcal{C}$ has fibre products. Let $X \in \mathrm{Ob}(\mathcal{C})$ be an object of $\mathcal{C}$. A hypercovering of X is a simplicial object K of $\mathrm{SR}(\mathcal{C}, X)$ such that*

(1) The object K_0 is a covering of X for the site $\mathcal{C}$.

(2) For every $n \geq 0$ the canonical morphism

$$K_{n+1} \longrightarrow (\mathrm{cosk}_n \, \mathrm{sk}_n \, K)_{n+1} \tag{7}$$

is a covering in the sense defined above.

Condition (1) makes sense since each object of $\mathrm{SR}(\mathcal{C}, X)$ is after all a family of morphisms with target X.

Remark IV.4. *By Lemma II.2,* $(\operatorname{cosk}_n \operatorname{sk}_n K)_{n+1} = \lim_{(\Delta/[n+1])^{op}_{\leq n}} \operatorname{sk}_n K(n+1)$. *Thus by the universal property of limits, the morphism (7) is given by a collection of morphisms* $\{\gamma(\alpha) : K_{n+1} \longrightarrow (\operatorname{sk}_n K)_k = K_k\}$, *where* α *ranges over* $\alpha : [k] \longrightarrow [n+1]$ *with* $k \leq n$.

Example IV.5. *(Čech hypercoverings). Let* $\mathcal{C}$ *be a site with fibre products. Let* $\{U_i \to X\}_{i\in I}$ *be a covering of* $\mathcal{C}$. *Set* $K_0 = \{U_i \to X\}_{i\in I}$. *Then* K_0 *is a* 0*-truncated simplicial object of* $\mathrm{SR}(\mathcal{C}, X)$. *Hence we may form*

$$K = \operatorname{cosk}_0 K_0$$

Clearly K *passes condition (1) of Definition IV.3. Since all the morphisms* $K_{n+1} \to (\operatorname{cosk}_n \operatorname{sk}_n K)_{n+1}$ *are isomorphisms by [Sta25, Tag 0AMA], Lemma 19.10, it also passes condition (2). Note that the terms* K_n *are*

$$K_n = (\operatorname{cosk}_0 K_0)_n = \lim_{(\Delta/[n])^{op}_{\leq 0}} K_0(n) = \prod_{j=0}^{n} K_0$$
$$= \{U_{i_0} \times_X U_{i_1} \times_X \ldots \times_X U_{i_n} \to X\}_{(i_0,i_1,\ldots,i_n)\in I^{n+1}}$$

A hypercovering of X *of this form is called a Čech hypercovering of* X.

Example IV.6. *(Hypercovering by a simplicial object of the site). Let* $\mathcal{C}$ *be a site with fibre products. Let* $X \in \mathrm{Ob}(\mathcal{C})$. *Let* U *be a simplicial object of* $\mathcal{C}$. *As usual we denote* $U_n = U([n])$. *Finally, assume given an augmentation*

$$a : U \to X,$$

i.e. a morphism from the simplicial object U *into the constant simplicial object* X. *In this situation we can consider the simplicial object* K *of* $\mathrm{SR}(\mathcal{C}, X)$ *with terms* $K_n = \{U_n \to X\}$. *Then* K *is a hypercovering of* X *in the sense of Definition IV.3 if and only if the following three conditions hold:*

(1) $\{U_0 \to X\}$ *is a covering of* $\mathcal{C}$,

(2) $\{U_1 \to U_0 \times_X U_0\}$ *is a covering of* $\mathcal{C}$,

(3) $\left\{U_{n+1} \to (\operatorname{cosk}_n \operatorname{sk}_n U)_{n+1}\right\}$ *is a covering of* $\mathcal{C}$ *for* $n \geq 1$.

We omit the straightforward verification. Note that as $\mathcal{C}$ *has fibre products, the category* $\mathcal{C}/X$ *has all finite limits. Hence the required coskeleta exist by Lemma II.2.*

Example IV.7. *(Čech hypercovering associated to a cover). Let* $\mathcal{C}$ *be a site with fibre products. Let* $U \to X$ *be a morphism of* $\mathcal{C}$ *such that* $\{U \to X\}$ *is a covering of* $\mathcal{C}$. *Consider the simplical object* K *of* $\mathrm{SR}(\mathcal{C}, X)$ *with terms*

$$K_n = \{U \times_X U \times_X \ldots \times_X U \to X\} \quad (n+1 \text{ factors })$$

Then K is a hypercovering of X. This example is a special case of both Example IV.5 and Example IV.6.

V Čech cohomology and hypercoverings

Let $\mathcal{C}$ be a site. For a presheaf of sets $\mathcal{F}$ we denote $\mathbf{Z}_{\mathcal{F}}$ the presheaf of abelian groups defined by the rule

$$\mathbf{Z}_{\mathcal{F}}(U) = \text{ free abelian group on } \mathcal{F}(U)$$

We will sometimes call this the free abelian presheaf on $\mathcal{F}$. Of course the construction $\mathcal{F} \mapsto \mathbf{Z}_{\mathcal{F}}$ is a functor and it is left adjoint to the forgetful functor $\mathrm{PAb}(\mathcal{C}) \to \mathrm{PSh}(\mathcal{C})$. Of course the sheafification $\mathbf{Z}_{\mathcal{F}}^{\#}$ is a sheaf of abelian groups, and the functor $\mathcal{F} \mapsto \mathbf{Z}_{\mathcal{F}}^{\#}$ is a left adjoint as well. We sometimes call $\mathbf{Z}_{\mathcal{F}}^{\#}$ the free abelian sheaf on $\mathcal{F}$.

For an object X of the site $\mathcal{C}$ we denote $\mathbf{Z}_X$ the free abelian presheaf on y_X, and we denote $\mathbf{Z}_X^{\#}$ its sheafification.

Definition V.1. *Let $\mathcal{C}$ be a site. Let K be a simplicial object of* $\mathrm{PSh}(\mathcal{C})$*. By the above we get a simplicial object $\mathbf{Z}_K^{\#}$ of* $\mathrm{Ab}(\mathcal{C})$*. We can take its alternating face map complex*[1] $C_\bullet \mathbf{Z}_K^{\#}$ *with $C_n \mathbf{Z}_K^{\#} := \mathbf{Z}_{K_n}^{\#}$. The homology of K is the homology of the complex of abelian sheaves $C_\bullet \mathbf{Z}_K^{\#}$.*

In other words, the ith homology $H_i(K)$ of K is the sheaf of abelian groups $H_i(K) = H_i(C_\bullet \mathbf{Z}_K^{\#})$. In this section we worry about the homology in case K is a hypercovering of an object X of $\mathcal{C}$.

Lemma V.2. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. The homology of the simplicial presheaf $F(K)$ is 0 in degrees* > 0 *and equal to $\mathbf{Z}_X^{\#}$ in degree 0 .*

Proof. See [Sta25, Tag 01GA] Lemma 4.5. □

Let $\mathcal{C}$ be a site. Consider a presheaf of abelian groups $\mathcal{F}$ on the site $\mathcal{C}$. It defines a functor

$$\begin{aligned} \mathcal{F} : \mathrm{SR}(\mathcal{C})^{op} &\longrightarrow \mathrm{Ab} \\ \{U_i\}_{i \in I} &\longmapsto \prod_{i \in I} \mathcal{F}(U_i) \end{aligned}$$

Thus a simplicial object K of $\mathrm{SR}(\mathcal{C})$ is turned into a cosimplicial object $\mathcal{F}(K)$ of Ab. The cochain complex $C_\bullet(\mathcal{F}(K))$ associated to $\mathcal{F}(K)$ is called the Čech complex of $\mathcal{F}$ with respect

[1] See Construction 1.3 of Lecture 10.

to the simplicial object K. We set

$$\check{H}^i(K, \mathcal{F}) = H^i(C_\bullet(\mathcal{F}(K))).$$

and we call it the ith Čech cohomology group of $\mathcal{F}$ with respect to K. In this section we prove analogues of some of the results for Čech cohomology of open coverings.

Lemma V.3. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathcal{C}$. Then $\check{H}^0(K, \mathcal{F}) = \mathcal{F}(X)$.*

Proof. Homework Exercise VII.2. □

Lemma V.4. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $\mathcal{I}$ be an injective sheaf of abelian groups on $\mathcal{C}$. Then*

$$\check{H}^p(K, \mathcal{I}) = \begin{cases} \mathcal{I}(X) & \text{if} \quad p = 0 \\ 0 & \text{if} \quad p > 0 \end{cases}$$

Proof. Observe that for any object $Z = \{U_i \to X\}$ of $\mathrm{SR}(\mathcal{C}, X)$ and any abelian sheaf $\mathcal{F}$ on $\mathcal{C}$ we have

$$\begin{aligned} \mathcal{F}(Z) &= \prod \mathcal{F}(U_i) \\ &= \prod \mathrm{Mor}_{PSh(\mathcal{C})}(\mathrm{y}_{U_i}, \mathcal{F}) \\ &= \mathrm{Mor}_{PSh(\mathcal{C})}(F(Z), \mathcal{F}) \\ &= \mathrm{Mor}_{PAb(\mathcal{C})}\left(\mathbf{Z}_{F(Z)}, \mathcal{F}\right) \\ &= \mathrm{Mor}_{Ab(\mathcal{C})}\left(\mathbf{Z}^\#_{F(Z)}, \mathcal{F}\right), \end{aligned}$$

where F is the functor defined in Definition I.2. Thus we see, for any simplicial object K of $\mathrm{SR}(\mathcal{C}, X)$ that we have

$$C_\bullet(\mathcal{F}(K)) = \mathrm{Hom}_{Ab(\mathcal{C})}\left(C_\bullet \mathbf{Z}^\#_{F(K)}, \mathcal{F}\right)$$

The complex of sheaves $C_\bullet \mathbf{Z}^\#_{F(K)}$ is quasi-isomorphic to $\mathbf{Z}^\#_X$ (the chain complex concentrated in degree 0) if K is a hypercovering, as implied by Lemma V.2. Since $\mathcal{I}$ is an injective abelian sheaf, $\mathrm{Hom}_{Ab(\mathcal{C})}(-, \mathcal{F})$ preserves quasi-isomorphism. Then the complex $C_\bullet(\mathcal{I}(K))$ is acyclic except possibly in degree 0 . In other words, we have

$$\check{H}^i(K, \mathcal{I}) = 0$$

for $i > 0$. Combined with Lemma V.3 the lemma is proved. □

VI Verdier Hypercovering Theorem

Lemma VI.1. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. If K, L are hypercoverings of X, then $K \times L$ is a hypercovering of X.*

Proof. This can be proved directly by Definition IV.3. Note that $(K \times L)_n = K_n \times_X L_n$. □

Lemma VI.2. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $k \geq 0$ be an integer. Let $u : Z \to K_k$ be a covering in $SR(\mathcal{C}, X)$. Then there exists a morphism of hypercoverings $f : L \to K$ such that $L_k \to K_k$ factors through u.*

Proof. See [Sta25, Tag 01GG], Lemma 7.3. □

Lemma VI.3. *Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K, L be hypercoverings of X. Let $a, b : K \to L$ be morphisms of hypercoverings. There exists a morphism of hypercoverings $c : K' \to K$ such that $a \circ c$ is homotopic to $b \circ c$.*

Proof. See [Sta25, Tag 01GO], Lemma 9.2. □

Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathcal{C}$. Let K, L be hypercoverings of X. If $a, b : K \to L$ are homotopic maps, then $\mathcal{F}(a), \mathcal{F}(b) : \mathcal{F}(K) \to \mathcal{F}(L)$ are homotopic maps ([Sta25, Tag 019U], Lemma 28.4). Hence they have the same effect on cohomology groups of the associated cochain complexes ([Sta25, Tag 019U], Lemma 28.6). We are going to use this to define the colimit over all hypercoverings.

Definition VI.4. *Denote $\mathrm{HC}(\mathcal{C}, X)$ the category whose objects are hypercoverings of X and whose morphisms are maps between hypercoverings of X up to homotopy. Consider the diagram*

$$\check{H}^i(-, \mathcal{F}) : \mathrm{HC}(\mathcal{C}, X)^{opp} \longrightarrow \mathrm{Ab}$$

and the ith hyper Čech cohomology group of $\mathcal{F}$ with respect to K is the colimit:

$$\check{H}^i_{\mathrm{HC}}(X, \mathcal{F}) = \mathrm{colim}_{K \in \mathrm{HC}(\mathcal{C}, X)} \check{H}^i(K, \mathcal{F})$$

Theorem VI.5. *(Verdier Hypercovering Theorem). Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let $i \geq 0$. The functors*

$$\begin{aligned} \mathrm{Ab}(\mathcal{C}) &\longrightarrow \mathrm{Ab} \\ \mathcal{F} &\longmapsto H^i(X, \mathcal{F}) \\ \mathcal{F} &\longmapsto \check{H}^i_{HC}(X, \mathcal{F}) \end{aligned}$$

are canonically isomorphic.

Proof. We have seen the result for $i = 0$, as implied by Lemma V.3. We also know that the hyper Čech cohomology is zero on injective sheaves by Lemma V.4. Since the functors $H^i(X, -)$ form a universal δ-functor, then in order to prove the theorem it suffices to show that the sequence of functors $\check{H}^i_{HC}(X, -)$ forms a δ-functor.

Let $0 \to \mathcal{F} \to \mathcal{G} \to \mathcal{H} \to 0$ be a short exact sequence of abelian sheaves on $\mathcal{C}$. Let $\xi \in \check{H}^p_{HC}(X, \mathcal{H})$. Choose a hypercovering K of X and an element $\sigma \in \mathcal{H}(K_p)$ representing ξ in cohomology. There is a corresponding exact sequence of complexes

$$0 \to C_\bullet(\mathcal{F}(K)) \to C_\bullet(\mathcal{G}(K)) \to C_\bullet(\mathcal{H}(K)),$$

but we are not assured that there is a zero on the right also and this is the only thing that prevents us from defining $\delta(\xi)$ by a simple application of the snake lemma. Recall that

$$\mathcal{H}(K_p) = \prod \mathcal{H}(U_i)$$

if $K_p = \{U_i \to X\}$. Let $\sigma = \prod \sigma_i$ with $\sigma_i \in \mathcal{H}(U_i)$. Since $\mathcal{G} \to \mathcal{H}$ is a surjection of sheaves we see that there exist coverings $\{U_{i,j} \to U_i\}$ such that $\sigma_i|_{U_{i,j}}$ is the image of some element $\tau_{i,j} \in \mathcal{G}(U_{i,j})$. Consider the object $Z = \{U_{i,j} \to X\}$ of the category $\mathrm{SR}(\mathcal{C}, X)$ and its obvious morphism $u : Z \to K_p$. It is clear that u is a covering. By Lemma VI.2 there exists a morphism $L \to K$ of hypercoverings of X such that $L_p \to K_p$ factors through u. After replacing K by L we may therefore assume that σ is the image of an element $\tau \in \mathcal{G}(K_p)$. Note that $d(\sigma) = 0$, but not necessarily $d(\tau) = 0$. Thus $d(\tau) \in \mathcal{F}(K_{p+1})$ is a cocycle. In this situation we define $\delta(\xi)$ as the class of the cocycle $d(\tau)$ in $\check{H}^{p+1}_{HC}(X, \mathcal{F})$.

At this point there are several things to verify:

(a) $\delta(\xi)$ does not depend on the choice of τ,

(b) $\delta(\xi)$ does not depend on the choice of the hypercovering $L \to K$ such that σ lifts, and

(c) $\delta(\xi)$ does not depend on the initial hypercovering and σ chosen to represent ξ.

We omit the verification of (a), (b), and (c); the independence of the choices of the hypercoverings really comes down to Lemma VI.1 and Lemma VI.3. We also omit the verification that δ is functorial with respect to morphisms of short exact sequences of abelian sheaves on $\mathcal{C}$.

Finally, we have to verify that with this definition of δ our short exact sequence of abelian sheaves above leads to a long exact sequence of Čech cohomology groups. First we show that if $\delta(\xi) = 0$ (with ξ as above) then ξ is the image of some element $\xi' \in \check{H}^p_{HC}(X, \mathcal{G})$. Namely, if $\delta(\xi) = 0$, then, with notation as above, we see that the class of $d(\tau)$ is zero in $\check{H}^{p+1}_{HC}(X, \mathcal{F})$. Hence there exists a morphism of hypercoverings $L \to K$ such that the restriction of $d(\tau)$ to an element of $\mathcal{F}(L_{p+1})$ is equal to $d(v)$ for some $v \in \mathcal{F}(L_p)$. This implies that $\tau|_{L_p} + v$ form a cocycle, and determine a class $\xi' \in \check{H}^p(L, \mathcal{G})$ which maps to ξ as desired.

We omit the proof that if $\xi' \in \check{H}^{p+1}_{HC}(X, \mathcal{F})$ maps to zero in $\check{H}^{p+1}_{HC}(X, \mathcal{G})$, then it is equal to $\delta(\xi)$ for some $\xi \in \check{H}^p_{HC}(X, \mathcal{H})$.

□

Note that there is another proof of the Verdier Hypercovering Theorem by using spectral sequences, see [Sta25, Tag 01GZ], Theorem 10.1.

VII Homework exercises

Exercise VII.1. *Prove Lemma IV.2:*

Let $\mathcal{C}$ be a site. (1) A composition of coverings in $SR(\mathcal{C})$ is a covering.

(2) If $\mathcal{C}$ has fibre products and $K \to L$ is a covering in $SR(\mathcal{C})$ and $L' \to L$ is a morphism, then $L' \times {}_L K$ exists and $L' \times_L K \to L'$ is a covering.

(3) If $\mathcal{C}$ has products of pairs, and $A \to B$ and $K \to L$ are coverings in $SR(\mathcal{C})$, then $A \times K \to B \times L$ is a covering.

Let $X \in \mathrm{Ob}(\mathcal{C})$. Then (1) and (2) holds for $SR(\mathcal{C}, X)$ and (3) holds if $\mathcal{C}$ has fibre products.

Exercise VII.2. *Prove Lemma V.3:*

Let $\mathcal{C}$ be a site with fibre products. Let X be an object of $\mathcal{C}$. Let K be a hypercovering of X. Let $\mathcal{F}$ be a sheaf of abelian groups on $\mathcal{C}$. Then $\check{H}^0(K, \mathcal{F}) = \mathcal{F}(X)$.

(Hint: The hypercovering condition requires that the canonical map $K_1 \to K_0 \times_X K_0$ is a covering in $\mathrm{SR}(\mathcal{C}, X)$. This means K_1 can be written as $K_1 = \coprod_{i_0, i_1 \in I} \coprod_{j \in J} \{V_{i_0 i_1 j} \to X\}$ where each $V_{i_0 i_1 j} \to U_{i_0} \times_X U_{i_1}$ is a covering in $\mathcal{C}$.)

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