Lecture ?(12) Today $H^1(\mathcal{C}, \mathcal{G})$ for $\mathcal{G}$ a sheaf of groups.

in AT we have $K(G, n)$, with the following two properties

- $\pi_i(K(G,n)) = 0$ or G ($i \neq n$, $i = n$)
- $[X, K(G,n)] = H^n(X, G)$

Cocycles, $(\mathcal{C}, \mathcal{J})$ $\mathcal{G}$ sheaf of grps. $\mathcal{U}$ a cover of an obj $X \in \mathcal{C}$.

$$\mathcal{U} = \{U_i \longrightarrow X\}$$

$U_{\alpha\beta} = U_\alpha \times_X U_\beta$ $\quad U_{\alpha\beta\gamma} = U_{\alpha\beta} \times_X U_\gamma$

A 1-cocycle is denoted as $\underline{g} := \{g_{ab} \in \mathcal{G}(U$

Satisfying the cocycle condition: $g_{ab}\, g_{bc} = g_{ac}$ ($\in U_{abc}$

two cocycles $\underline{g}, \underline{h}$ are equiv. if $\exists \{f_a\} \in \mathcal{G}(U_a)\}$ Such that

$$f_a\, g_{ab} = h_{ab}\, f_b$$

We define $\check{H}^1(\mathcal{U}, \mathcal{G})$ to be $\dfrac{\text{1-cocycles}}{\text{equivalences}}$.

Finally we define $\check{H}^1(X, \mathcal{G}) = \operatorname{colim}_{\mathcal{U}} \check{H}^1(\mathcal{U}, \mathcal{G})$.

Note that this looks very much like Čech!

(And actually, it is). Recall that the Čech cmpx ~~cohom.~~ is defined by

$$\prod_a \mathcal{G}(U_a) \xrightarrow{d_0 - d_1} \prod_{a,b} \mathcal{G}(U_{ab}) \xrightarrow{d_0 - d_1 + d_2} \prod_{abc} \mathcal{G}(U_{abc})$$

$$f_a, f_b \longmapsto f_a f_b^{-1} \in U_{ab}$$

$$\begin{matrix} f_{ab} \\ f_{bc} \\ f_{ac} \end{matrix} \longmapsto f_{ab} f_{bc} f_{ac}^{-1}$$

This shows that whenever $\mathcal{G}$ is a sheaf of ab. grps. our two notions coincide. And as we have seen in Čech cohom this is iso to Sheaf ☺ Cohomology (we saw Čech is iso in $n=0,1$)

What is $H^1(\mathcal{U}, \mathcal{G})$? Not a group
why? (Just a ptd set, point → neutral element)

these are Not grp homomorphisms.
notice $\{f_a g_a\} \longrightarrow \{f_a g_a (f_b g_b)^{-1}$

$$g_b^{-1} f_b^{-1}$$

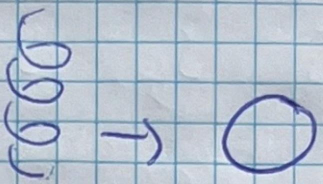
$$f_a f_b^{-1} \neq g_a g_b^{-1}$$

Étale Spaces

Fix a space X, an Étale space over X is a space E which locally looks like a homeomorphism, ~~onto~~ A morphism of étale spaces is a map of spaces which commutes w.r.t. X.

$$\begin{tikzcd} E \arrow[r] \arrow[dr] & E' \arrow[d] \\ & X \end{tikzcd}$$

Examples:



$\mathbb{R} \to S^1$

Fibre bundles over X.

$$\text{Étale} \underset{\leftrightarrows}{\cong} \text{Sheaves}$$

$\to$ Sections

$\leftarrow$ All stalks

$\to$ Define $S(U)$ to be sections ~~on~~ $X \to E$ ~~[illegible]~~ (this is clearly a sheaf?)

$\leftarrow$ Given sheaf P on X, define P_x ~~[illegible]~~ colim $P(U)$

Now the ~~set~~ set $E(P) = \coprod_{x \in X} P_x$ (with projection to X)

with topology basis defined as $\{ \text{germ}_x(a) \mid x \in U \}$, $a \in P(U)$ $(\text{germ}_x : P(U) \to P_x)$

is an étale space.

[illegible] Étale space of groups $\begin{array}{c} G \\ \downarrow \\ X \end{array}$

unit $X \to G$

m: $G \times_X G \to G$

inv: G

First Étale Spaces, Remember Torsors Continued!!!

Torsors

Again, let $\mathcal{G}$ be a sheaf of Grps and let $\mathcal{S}$ be any sheaf.

$\mathcal{S}$ is a $\mathcal{G}$-torsor if there exist a map sheaves $\mu : \mathcal{G} \times \mathcal{S} \to \mathcal{S}$ (so this locally it is a (left) action on each $\mathcal{S}(U)$, working well with all restr. etc.)

+ $\exists$ covering $\{U_\alpha \to X\}$ of X such that $\mathcal{S}(U_\alpha) \neq \emptyset$
+ $(\mu, \pi_2) : G \times S \longrightarrow S \times S$ is an iso.

A morphism of G-torsors is a morphism of sheaves which commute with the action.

On spaces X

+ $X = \bigcup \{U \mid S(U) \neq \emptyset\}$ "there exist a cover with non empty sections"
+ the action μ on $S(U)$ is free + transitive

Then on an étale space

+ $\mu : G \times_X E \to E$
+ $E \to X$ is surjective
+ (μ, π_2) $G \times_X E \longrightarrow E \times_X E$ Homeo

Torsors ~~⊗~~ cont.

$(\mu,\pi_2): \mathcal{G}\times S \to S\times S$ iso, $\Longrightarrow$ that

$\forall U$ ~~$\mathcal{G}$~~ $\mathcal{G}(U)\times S(U) \to S(U)\times S(U)$ is a iso

This the same as saying ~~that~~ the action is free and a transitive.

Interpretation as étale space:

~~$\mathcal{G}$~~

$G \to X$

Étale space with group ~~action~~ mult etc.

$G\times_X G \to G$

$\mu: G\times_X E \to E$

$+\ E \longrightarrow X$ surj.

$+\ G\times_X E \longrightarrow E\times_X E$ is iso

What is $E \to X$ now?

A principal G-Bundle!

Torsors $\Longleftrightarrow$ Cocycles

"$\longrightarrow$"

Let S $\mathcal{G}$-torsor over X,

(1) thus $\exists$ cover $X = \bigcup U_\alpha$ with sections in $s_\alpha \in S(U_\alpha)$

(2) $\forall$ s_α, s_β $\exists$ $g_{\alpha\beta}$ such that

$$g_{\alpha\beta} \cdot s_\beta = s_\alpha \quad (\in U_{\alpha\beta})$$

this defines a cocycle in $\mathcal{G}$,

We verify

$$g_{\alpha\beta} \cdot s_\beta = s_\alpha$$
$$g_{\beta\gamma} \cdot s_\gamma = s_\beta$$
$$g_{\alpha\gamma} \cdot s_\gamma = s_\alpha$$
$$\Downarrow$$
$$= g_{\alpha\beta}\, s_\beta = g_{\alpha\beta}\, g_{\beta\gamma}\, s_\gamma$$
$$\Downarrow$$
$$g_{\alpha\gamma} = g_{\alpha\beta}\, g_{\beta\gamma} \quad \text{(free action)}$$

Now we show this is independent of our choice s_α. Assume we had chosen t_α, then $\exists f_\alpha \cdot s_\alpha = t_\alpha$

$\hookrightarrow$ this gives us the cocycle $h_{\alpha\beta}$.

$$f_\alpha \cdot s_\alpha = t_\alpha$$
$$\Downarrow$$
$$f_\alpha\, g_{\alpha\beta}\, s_\beta = h_{\alpha\beta}\, t_\beta = h_{\alpha\beta}\, f_\beta\, s_\beta$$
$$\Downarrow \text{(free)}$$
$$f_\alpha\, g_{\alpha\beta} = h_{\alpha\beta}\, f_\beta$$

thus choice of sections is independent of eq. class of cocycle

this gives us a class in $H^1(\{U_\alpha\}, \mathcal{G})$, and varying the cover we get a class in $\check{H}^1(X, \mathcal{G})$

""

Let $\{g_{\alpha\beta}\}$ be a cocycle wrt. a cover $\{U_\alpha \to X\}$

We ~~define~~ Define the sheaf S as follows

Locally, meaning on the U_α, $S(U_\alpha) = \mathcal{G}(U_\alpha)$

Now we let $S(V) := \{ (s_\alpha)_\alpha \in \prod_\alpha \mathcal{G}(U_\alpha \times_X V)$

$\mid g_{\alpha\beta} s_\alpha = s_\beta \}$

Note that this is invariant under equivalence of torsors, $(g_{\alpha\beta} s_\alpha = h_{\alpha\beta} f_\beta s_\alpha = f_\alpha^{-1} s_\beta)$

Meaning, it merely permutes, but the "same" elements are still in there.

We omit the verification that this is a sheaf and the details for the restriction maps.

- So note we have a clear left action of $\mathcal{G}$ (as elements are just a product of $\mathcal{G}(U_\alpha)$'s).
- S has a nonempty cover
- S looks like $\mathcal{G}$

Torsors $\leftrightarrow$ Cocycles

Thm: These operations are each others inverses (up to isomorphism) of $\mathcal{G}$-torsors, and equivalences of $\check{H}^1(X, \mathcal{G})$

this gives us a nice proof that Prin-G-Bundles are fully defined by $\check{H}^1$.

One is now happy, as we have a geometric interpretation of $\check{H}^1$

Exposition about $\mathcal{B}G$

Like in AT, there exists a notion of classifying space for topoi (named classifying topoi).

in AT, BG is not so easy to construct. But we have $[X, BG] \cong H^1(X; G) \cong \mathrm{Prin}_G(X)$

There is (in topoi land) also a $\mathcal{B}G$ such that $[Sh(X), \mathcal{B}G] \cong \mathrm{Prin}_G(X)$

but here, $\mathcal{B}G$ is really easy to construct, as $\mathcal{B}G \cong PSh(G^{op})$ (Group seen as a Grpd)

How would a $\mathcal{B}^2G$ look like now?

Delooping once was putting G in the morphisms

Delooping twice means we should put G in the two-morphisms. So:

$$\begin{array}{cc} G & C_2 \\ \bullet & C_1 \\ \bullet & C_0 \end{array}$$

this is now a 2-Cat, and we can ~~interpret~~

(this only works when G is abelian)

look at the 2-topos of pseudo functors from

(We can think of this as $K(G,2)$) $[G^{op}, \text{Grpd}]_{\text{Pseudo}}$ $\quad G^{op} \to$ Grpd

~~This now requires some language of higher cat theory~~

Now for any 1-topos X we have that

$$[X, \beta\beta G] \cong H^2(X, \underline{G})$$

Higher Category Stuff

Fibered Categories $\mathbb{F}$ over $\underset{\text{1-Cat}}{\mathcal{C}}$

- $\forall\, c \in \mathcal{C}_0$ a category $\mathbb{F}(c)$
- $\forall\, i : c \to d$ a functor $i^* : \mathbb{F}(d) \to \mathbb{F}(c)$
- $\forall\, i,j$ a T_{ij}, natural isomorphism from

$$(ij)^* \to j^* i^*$$

~~Ex~~

Examples: – Any presheaf of categories
(a functor from $C^{op} \to Cat$)

Note because it is a functor, the T_{ij} become equalities. So a fibered category is one where we allow the composition to be defined up to isomorphism.

~~– X top space. $U \mapsto Fib(U)$ the functor defined by pullbacks ... only defined up to iso ... bundles ...~~

– X top space, we define $U \longmapsto Fib(U)$ where the functor between $U' \to U$,
$Fib(U) \to Fib(U')$
is defined as the pullback fiber bundle.

– (this probably works for cat of all top spaces

Stack

– X top space, ~~the assign~~ $U \longmapsto Sheaves(U)$ is a stack.

– fix sheaf of grp $\mathcal{G}$.
$U \longmapsto$ Torsors over $\mathcal{G}|_U$ form a stack.

Prestacks

For $c \in \mathcal{C}$ and $a, b \in \mathbb{F}(c)$ we can look at the presheaf:

$$c' \longmapsto \mathrm{Hom}_{\mathbb{F}(c')}(i^*a, i^*b), \quad \text{for } i : c' \to c$$

We denote this presheaf as $\underline{\mathrm{Hom}}_{\mathbb{F}(c)}(a, b)$.

Note that this is an element of $[(\mathcal{C}/_c)^{op}, Set]$.

This can be done for any fibered category.
And if for all c, a, b this is a <u>sheaf</u>.
Then ~~we call~~ it is a prestack.

"local isomorphic objects are isomorphic"

<u>Example</u>. Fix a presheaf P of sets

Any separated presheaf (viewed as a ~~prestack~~ fibered category) is a prestack. <u>HW</u>

Descent Data. Fix a fibered Cat $\mathbb{F}$, over $\mathcal{C}$

Let $\{U_\alpha \to X\}$ be a cover. We define a Cat $Des(\{U_\alpha\}, \mathbb{F})$

- Objects: $(\{a_\alpha\}, \{\Theta_{\alpha\beta}\})$ with $a_\alpha \in \mathbb{F}(U_\alpha)$

  $\Theta_{\alpha\beta} : a_{\alpha\beta} \cong a_{\alpha}$ (in $\mathbb{F}(U_{\alpha\beta})$)

  such that $\Theta_{\alpha\alpha} = id$, $\Theta_{\alpha\beta} \circ \Theta_{\beta\gamma} = \Theta_{\alpha\gamma}$ (in $\mathbb{F}(U_{\alpha\beta\gamma})$)

- Morphisms, $f : (a, \Theta) \xrightarrow{f} (b, \rho)$ is $f_\alpha : a_\alpha \to b_\alpha$ st.

$$\begin{tikzcd} a_\beta \arrow[r, "f_\beta"] \arrow[d, "\Theta_{\alpha\beta}"'] & b_\beta \arrow[d, "\rho_{\alpha\beta}"] \\ a_\alpha \arrow[r, "f_\alpha"'] & b_\alpha \end{tikzcd}$$

commutes

There exists a functor D from $\mathbb{F}(X) \to Des(\{U_\alpha\}, \mathbb{F})$

Prop which is fully faithful iff $\mathbb{F}$ is a prestack

Def if D is an equivalence of Cats then we call $\mathbb{F}$ a stack.

Prop A prestack is stack iff for any covers, any object of $Des(\mathcal{U}, \mathbb{F})$ is isomorphic to $D(b)$ (for some $b \in \mathbb{F}(X)$) (Examples!)

"Objects which we can glue locally form a unique global object"