Automorphism Sheaves

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I made some mistakes during the lecture and it seems to have caused some confusion. Therefore, I would like to clarify some issues relating to the automorphism sheaf.

Let $\mathrm{Aut}_{\mathrm{Sh}}(\mathcal{G})$ denote the set of automorphisms of *sheaves* of groups, $\mathrm{Aut}_{\mathrm{Grp}}(G)$ the set of automorphisms of groups and $\underline{\mathrm{Aut}}(\mathcal{G})$ the automorphism sheaf.

The bottom line is:

$$\underline{\mathrm{Aut}}(\mathcal{G})(U) = \mathrm{Aut}_{\mathrm{Sh}}(\mathcal{G}|_U)$$

without any sheafification. Note that the set $\mathrm{Aut}_{\mathrm{Grp}}(G)$, shouldn't really appear anywhere. To obtain the outer automorphisms one would need to take a quotient, which is a colimit and hence sheafification is involved. So the outer automorphism sheaf is the sheafification of $U \mapsto \underline{\mathrm{Aut}}(\mathcal{G})(U)/\mathcal{G}(U)$.

How to derive this stuff?

To construct the automorphism sheaf we would like to internalize group theory in the topos of sheaves of sets. Therefore, the sheaf $\underline{\mathrm{Aut}}(\mathcal{G})$ should be some subsheaf of the exponential sheaf $\mathcal{G}^{\mathcal{G}}$.

To construct the exponential sheaf, we use the yoneda lemma. We have

$$\begin{aligned}
\mathcal{G}^{\mathcal{F}}(U) &\cong \mathrm{Hom}_{\mathrm{PSh}}(\mathrm{Y}_U, \mathcal{G}^{\mathcal{F}}) && \text{Yoneda}\\
&\cong \mathrm{Hom}_{\mathrm{Sh}}(\mathrm{Y}_U^{\#}, \mathcal{G}^{\mathcal{F}}) && \text{Sheafification adjunction}\\
&\cong \mathrm{Hom}_{\mathrm{Sh}}(\mathrm{Y}_U^{\#} \times \mathcal{F}, \mathcal{G}), && \text{Exponential adjunction}
\end{aligned}$$

hence we define

$$\mathcal{G}^{\mathcal{F}}(U) = \mathrm{Hom}_{\mathrm{Sh}}(\mathrm{Y}_U^{\#} \times \mathcal{F}, \mathcal{G}).$$

Note that in our case Y_U is just an indicator function

$$Y_U(V) = \begin{cases} 1 & V \subseteq U \\ 0 & V \nsubseteq U. \end{cases}$$

This is in particular, already a sheaf. This is because if we have sections $s_\alpha \in Y_U(V_\alpha)$ then $V_\alpha \subseteq U$ and hence for $V = \bigcup_\alpha V_\alpha$ we have $V \subseteq U$, which implies that there is some section $s \in Y_V(U)$. Therefore

$$\mathcal{G}^{\mathcal{F}}(U) = \mathrm{Hom}_{\mathrm{Sh}}(\mathrm{Y}_U \times \mathcal{F}, \mathcal{G}) \cong \mathrm{Hom}_{\mathrm{Sh}}(\mathcal{F}|_U, \mathcal{G}|_U),$$

where this last isomorphism follows from the fact that $Y_U(V)$ simply witnesses $V \subseteq U$.

With this explicit description of the exponential sheaf it is not hard to see that the limit construction that carves out internal group automorphisms is preserved by the inclusion of sheaves into presheaves and hence can be performed objectwise. We obtain indeed

$$\underline{\mathrm{Aut}}(\mathcal{G})(U) = \mathrm{Aut}_{\mathrm{Sh}}(\mathcal{G}|_U).$$