LAX FUNCTORIALITIES OF THE COMMA CONSTRUCTION FOR ω-CATEGORIES

by

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1. Introduction

***1.1* (Motivation).** — The classical Grothendieck construction defines, for every small category I, a functor

$$\textstyle\int_I : \underline{\mathsf{Hom}}(I^{\mathrm{op}}, \mathcal{C}at) \to \mathcal{C}at$$

that sends a functor $F : I^{\mathrm{op}} \to \mathcal{C}at$, from I^{op} to the category of small categories $\mathcal{C}at$, to the so-called *Grothendieck construction* $\int_I F$ of F. Here $\underline{\mathsf{Hom}}$ denotes the cartesian internal Hom of $\mathcal{C}at$, whose morphisms are strict natural transformations. But the functorialities of the Grothendieck construction are more general. First, if $F, G : I^{\mathrm{op}} \to \mathcal{C}at$ are two such functors and $\alpha : F \Rightarrow G$ is a *oplax* transformation (that is, roughly speaking, a transformation where the naturality squares only commute up to an oriented 2-cell), then one can still integrate α to obtain a functor $\int_I \alpha : \int_I F \to \int_I G$. Second, the construction is also functorial in I. Combining these, we get a functoriality

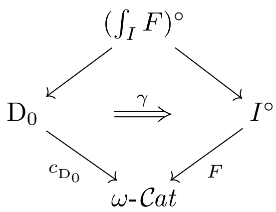
$$\begin{array}{ccc} I^{\mathrm{op}} & \xrightarrow{u^{\mathrm{op}}} & J^{\mathrm{op}} \\ & {}_{F}\searrow \overset{\alpha}{\Rrightarrow} \swarrow_{G} & \\ & \mathcal{C}at & \end{array} \qquad \mapsto \qquad \textstyle\int_I F \xrightarrow{\int(u,\alpha)} \int_J G \quad ,$$

where α is an oplax transformation.

The purpose of this paper is to study higher generalizations of these functorialities in the setting of strict ω-categories. Our original motivation was to investigate the homotopical properties of the Grothendieck construction for strict ω-categories, particularly the generalization of a theorem by Thomason [**7**], which will be addressed in a separate paper [**2**].

1.2 **(ω-categorical comma construction).** — Let $\omega\text{-}\mathcal{C}at$ denote the ω-category of strict ω-categories (with the cartesian enrichment). If $F : I^\circ \to \omega\text{-}\mathcal{C}at$ is a strict ω-functor, where I is a strict ω-category and I° is the dual obtained by reversing the orientation of all the cells, then a Grothendieck construction $\int_I F$ was defined by Warren in his work on the model of strict ω-groupoids for dependent type theory [**8**].

However, Warren's definition is unsatisfactory as it relies on explicit and complicated formulas. We propose to *define* the Grothendieck construction of $F : I^\circ \to \omega\text{-}\mathcal{C}at$ as the ω-category $\int_I F$ endowed with a universal 2-square



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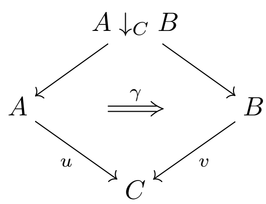
where D_0 denotes the terminal ω-category, c_{D_0} the constant ω-functor of value D_0, and γ an oplax transformation. This type of universal 2-squares was already studied by the first-named author and Maltsiniotis [**3**] and is a straightforward generalization of the classical comma construction, usually denoted $u \downarrow v$. More precisely, we have

$$\int_I F = (c_{D_0} \downarrow F)^\circ ,$$

where $\downarrow$ denotes the *oplax* comma construction. Although these definitions are abstract, explicit formulas can be extracted, and we recover Warren's formulas from this abstract point of view.

We are thus led to consider the following more general case. Let A, B, C be three strict ω-categories and $u\colon A \to C$ and $v\colon B \to C$ be two ω-functors. We can form the (oplax) comma ω-category $A \downarrow_C B$, which comes equipped with a universal 2-square:



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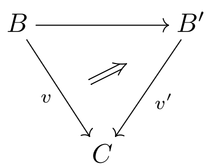
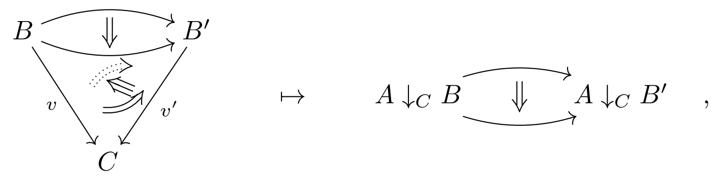
where γ is a oplax transformation. Our goal is now to study the functorialies of $A \downarrow_C B$ in $v\colon B \to C$, with C fixed, and symetrically in $u\colon A \to C$, for which the functorialities of the Grothendieck construction are a particular case. The universal property of the comma immediately gives a functoriality:

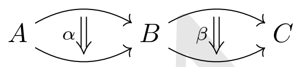


$\mapsto \qquad A \downarrow_C B \longrightarrow A \downarrow_C B'$, where the 2-cell represents an oplax transformation. Working a bit harder, one can get a functoriality:



where the 2-cells represent oplax transformations and the 3-cell represents an oplax 2-transformation (also known as a oplax modification). And now comes the question: what is the general statement?

***1.3* (Slices of Gray ω-categories).** — The answer to this question uses the language of Gray ω-categories, which the first-named author introduced with Maltsiniotis in their work on the join construction and the slices [**4**]. Indeed, the diagrams above involving 0-cells, 1-cells, 2-cells and 3-cells actually live in $\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$, in which 0-cells are strict ω-categories, 1-cells are strict ω-functors, 2-cells are oplax transformations, 3-cells are oplax 2-transformations, and so on. But $\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$ is not an ω-category, *not even weak*! Indeed, if



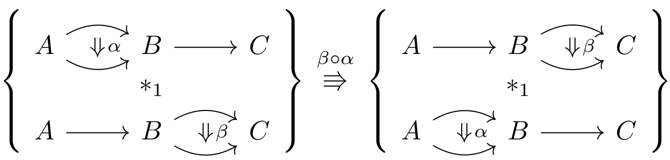
are two oplax transformations, then there are a priori two ways of composing them:

$$(t(\beta) *_0 \alpha) *_1 (\beta *_0 s(\alpha)) \quad \text{and} \quad (\beta *_0 t(\alpha)) *_1 (s(\beta) *_0 \alpha)\,,$$

where s and t denote the source and the target. In general, these two oplax transformations are different! In other words, $\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$ does not satisfy the exchange rule. What is true is that there is a **non-invertible** canonical oplax 2-transformation:

$$(\beta *_0 t(\alpha)) *_1 (s(\beta) *_0 \alpha) \overset{\beta\circ\alpha}{\Longrightarrow} (t(\beta) *_0 \alpha)) *_1 (\beta *_0 s(\alpha))\,,$$

which can be pictured as:



This means that $\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$ is some kind of oplax ω-category. Formally, $\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$ is what we call a *Gray ω-category*, that is, a category enriched in $\omega\text{-}\mathcal{C}at$ endowed with the oplax Gray tensor product. Morphisms of Gray ω-categories are called *Gray ω-functors.*

It is now tempting to think that the correspondence:

$$A \downarrow_C - \colon (B \to C) \mapsto A \downarrow_C B$$

extends to a Gray ω-functor of target $\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$. But what would be the source Gray ω-category? Or, in other words, in which Gray ω-category do the triangles and cones we drew earlier are 1-cells and 2-cells? Obviously, in some kind of slice Gray ω-category $\omega\text{-}\mathbb{C}\mathsf{at}_{\text{oplax}}/C$ of the Gray ω-category $\omega\text{-}\mathbb{C}\mathsf{at}_{\text{oplax}}$ above the strict ω-category C (which is an object of $\omega\text{-}\mathbb{C}\mathsf{at}_{\text{oplax}}$). More generally, we prove the following, which formalizes the notion of slice of a Gray ω-category.

Proposition 1.4. — *Let $\mathbb{C}$ is a Gray ω-category and c is an object of $\mathbb{C}$. Then there is a natural Gray ω-category $\mathbb{C}/_c$ of objects of $\mathbb{C}$ over c. In the case that $\mathbb{C}$ is a* strict *ω-category (which we can consider as a Gray ω-category where the "interchange rule" is an equality), we recover the usual notion of slice strict ω-category* [**4**].

The existence of slices of Gray ω-categories was conjectured in [**4**, Conjecture C.24].

***1.5* (Functoriality results).** — Using slices of Gray ω-categories, we can finally express the desired functoriality of the construction $A \downarrow_C B$, actually both in A and B simultaneously.

Theorem 1.6. — *The oplax comma construction $- \downarrow_C -$ defines a Gray ω-functor*

$$- \downarrow_C - : \omega\text{-}\mathbb{C}\mathsf{at}_{\text{oplax}}/C \times \omega\text{-}\mathbb{C}\mathsf{at}_{\text{oplax}}\overset{\text{to}}{/}C \to \omega\text{-}\mathbb{C}\mathsf{at}_{\text{oplax}}\,.$$

The decoration "to" in the previous theorem is a particular duality of slices of Gray ω-category. As it happens, dualities of (strict and Gray) ω-categories play an important role in this paper, and we study them with great care.

We also studied restricted functorialities of the comma construction. The ω-category $\omega\text{-}\mathcal{C}at$ of ω-categories can be seen as sub Gray ω-category:

$$\omega\text{-}\mathcal{C}at \hookrightarrow \omega\text{-}\mathbb{C}\mathsf{at}_{\text{oplax}},$$

by considering every strict higher transformation as a particular case of higher oplax transformations. We then proved the following result.

Proposition 1.7. — *The oplax comma construction restricts to a (strict) ω-functor*

$$- \downarrow_C -: \omega\text{-}\mathcal{C}at/C \times \omega\text{-}\mathcal{C}at\overset{\text{to}}{/}C \to \omega\text{-}\mathcal{C}at.$$

The non-trivial part of this proposition being that the target of this ω-functor does indeed restrict to $\omega\text{-}\mathcal{C}at$. Once these functoralities are proven, we can finally study the functorialities of the Grothendieck construction. For that, notice that the ω-category $\omega\text{-}\mathcal{C}at$ is an object of $\omega\text{-}\mathbb{C}\mathrm{AT}_{\text{oplax}}$ the (very large) Gray ω-category of large ω-categories, ω-functors and lax (higher) transformations between them.

Corollary 1.8. — *The Grothendieck construction defines a Gray ω-functor:*

$$\int : \left(\omega\text{-}\mathbb{C}\mathsf{at}_{\text{oplax}}\overset{\text{to}}{/}\omega\text{-}\mathcal{C}at\right)^{\text{to}} \to \omega\text{-}\mathbb{C}\mathsf{at}_{\text{oplax}}$$

$$(F\colon I^\circ \to \omega\text{-}\mathcal{C}at) \mapsto \int_I F,$$

where $\omega\text{-}\mathbb{C}\mathsf{at}_{\text{oplax}}\overset{\text{to}}{/}\omega\text{-}\mathcal{C}at$ is the full sub-Gray ω-category of $\omega\text{-}\mathbb{C}\mathrm{AT}_{\text{oplax}}\overset{\text{to}}{/}\omega\text{-}\mathcal{C}at$ spanned by those functors $F\colon I^\circ \to \omega\text{-}\mathcal{C}at$, where I is a small ω-category.

In particular, if we fix a (small) ω-category I, the above restricts to a Gray ω-functor

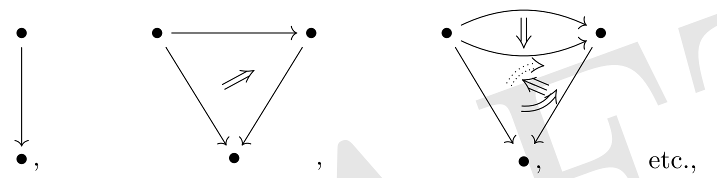
$$\int_I \colon \underline{\mathsf{Hom}}_{\mathrm{oplax}}(I^\circ, \omega\text{-}\mathcal{C}at) \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\,,$$

as well as a strict ω-functor

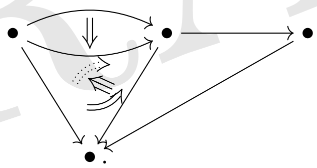
$$\int_I \colon \underline{\mathsf{Hom}}(I^\circ, \omega\text{-}\mathcal{C}at) \to \omega\text{-}\mathcal{C}at,$$

where $\underline{\mathsf{Hom}}$ is the cartesian internal Hom of ω-$\mathcal{C}at$.

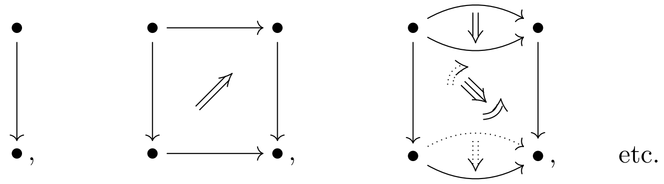
***1.9* (Cones and other shapes).** — It seems important to bring the reader's attention to the fact that there is a *tour de force* behind the definition of slices Gray ω-category $\mathbb{C}_{/c}$. Intuitively, to define such Gray ω-categories, one must make sense in an arbitrary Gray ω-category of pasting diagrams shaped like (higher) cones:



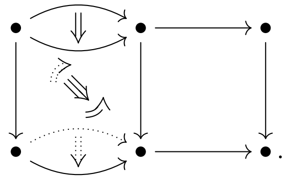
as well as defining compositions between those. For example, defining the whiskering operation of a 2-cell with a 1-cell in slice Gray ω-categories amounts to define a "total composite" of the following pasting diagram



In the case of *strict* ω-categories, the "cone"-shaped pasting diagrams are considered as degenerate cases of "cylinder" shaped pasting diagrams:



and the composition between cone-shaped diagrams is induced by the ones at the level of cylinders. *However,* it seems that for Gray ω-categories, it is not possible to define all the cylinder compositions that would be required to then deduce the ones for cones as a particular case. For example, we let the reader convince themselves that it is not possible to make sense of the total composite of the following pasting diagram in a Gray ω-category:



***1.10* (A generalization).** — Let us end this introduction with a noticeable by-product result that we obtained in this paper. As noted before, the Grothendieck construction of an ω-functor $F\colon I^\circ \to \omega\text{-}\mathcal{C}at$ can be defined as (the dual of) a comma-ω-category. In fact, this comma-ω-category is (the dual of) a relative slice, meaning it is obtained by pulling back a slice ω-category as follows

$$\begin{array}{ccc} (\int_I F)^\circ & \longrightarrow & \mathrm{D}_0\backslash\omega\text{-}\mathcal{C}at \\ \big\downarrow & \lrcorner & \big\downarrow \\ I^\circ & \xrightarrow[F]{} & \omega\text{-}\mathcal{C}at. \end{array}$$

The advantage of this description is that it can be adapted straightforwardly in the context of Gray ω-categories, up to some subtleties on dualities. If $\mathbb{C}$ be a (small) Gray ω-category, then its "total dual" obtained by reversing the directions of all the cells is not a Gray category but what we call a *skew* Gray ω-category. By that, we mean a category enriched in $\omega\text{-}\mathcal{C}at$, endowed with the oplax Gray tensor product, but with the "type" of composition morphisms given by

$$\underline{\mathsf{Hom}}(X,Y)\otimes\mathsf{Hom}(Y,Z)\to\mathsf{Hom}(X,Z)$$

(as opposed to $\mathsf{Hom}(Y,Z)\otimes\mathsf{Hom}(X,Y)\to\mathsf{Hom}(X,Z)$, which is a very different notion as the oplax Gray tensor product is not symetrical or even braided). As an example of skew Gray ω-category we have $\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}$, whose 0-cells are strict ω-categories, 1-cells are ω-functors and higher cells are higher *lax* transformations. Now, given a (small) Gray ω-category $\mathbb{C}$ and $F\colon \mathbb{C}^\circ\to\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}$ a skew Gray ω-functor, we define the (dual of the) Grothendieck construction $\int_{\mathbb{C}} F$ of F as the following pullback:

$$\begin{array}{ccc} (\int_{\mathbb{C}} F)^\circ & \longrightarrow & \mathrm{D}_0\backslash\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}} \\ \big\downarrow & \lrcorner & \big\downarrow \\ \mathbb{C}^\circ & \xrightarrow[F]{} & \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}\,. \end{array}$$

Note that $\int_{\mathbb{C}} F$ is indeed a Gray ω-category (and not a skew Gray ω-category). As already said, the dualities for strict and Gray ω-categories are subtle and we give study them thoroughly in this paper.

We plan to work more extensively on this Grothendieck construction in the context of Gray ω-categories in future work.

2. Preliminaries on enriched categories

We start by some preliminaries on categories enriched in a monoidal category. Our case of interest is the category of strict ω-categories endowed with the Gray tensor product, which will be introduced in the next section. This tensor product is *not* symmetric (nor even braided).

2.1. — Let $\mathcal{V} = (\mathcal{V}, \otimes, I)$ be a monoidal category. Since we do not assume $\mathcal{V}$ to be symmetric, we need to distinguish between the notion of a $\mathcal{V}$-category (or a category enriched in $\mathcal{V}$) and that of a *skew* $\mathcal{V}$-category.

A *$\mathcal{V}$-category* $\mathcal{A}$ is given by

- a set of *objects* $\mathsf{Ob}(\mathcal{A})$,
- for every objects X and Y of $\mathcal{A}$, an *object of morphisms* $\underline{\mathsf{Hom}}_{\mathcal{A}}(X, Y)$ in $\mathcal{V}$,
- for every objects X, Y and Z of $\mathcal{A}$, a *composition morphism*
$$\circ\colon \underline{\mathsf{Hom}}_{\mathcal{A}}(Y, Z) \otimes \underline{\mathsf{Hom}}_{\mathcal{A}}(X, Y) \to \underline{\mathsf{Hom}}_{\mathcal{A}}(X, Z)$$
in $\mathcal{V}$,
- for every object X of $\mathcal{A}$, an *identity morphism*
$$1_X\colon I \to \underline{\mathsf{Hom}}_{\mathcal{A}}(X, X)\,.$$
in $\mathcal{V}$,

satisfying well-known axioms.

The notion of a *skew* $\mathcal{V}$-category is obtained likewise but with composition morphisms of type
$$\circ\colon \underline{\mathsf{Hom}}_{\mathcal{A}}(X, Y) \otimes \underline{\mathsf{Hom}}_{\mathcal{A}}(Y, Z) \to \underline{\mathsf{Hom}}_{\mathcal{A}}(X, Z)\,.$$
More formally, a skew $\mathcal{V}$-category is a $\overline{\mathcal{V}}$-category, where $\overline{\mathcal{V}}$ denotes the monoidal category $(\mathcal{V}, \overline{\otimes}, I)$, the monoidal product $\overline{\otimes}$ being defined by $X \mathbin{\overline{\otimes}} Y := Y \otimes X$.

We denote by $\mathcal{V}\text{-}\mathcal{C}at$ the (large) 2-category whose objects are $\mathcal{V}$-categories, whose morphisms are $\mathcal{V}$-functors and whose 2-morphisms are $\mathcal{V}$-natural transformations. The 2-category $\overline{\mathcal{V}}\text{-}\mathcal{C}at$ is thus the 2-category of skew $\mathcal{V}$-categories. Its morphisms are called *skew $\mathcal{V}$-functors* and its 2-morphisms *skew $\mathcal{V}$-natural transformations.*

2.2. — Let $\mathcal{V}$ be a monoidal category. Given a $\mathcal{V}$-category $\mathcal{A}$, we define its *transpose* to be the obvious *skew* $\mathcal{V}$-category $\mathcal{A}^{\mathrm{t}}$ with same set of objects as $\mathcal{A}$ and
$$\underline{\mathsf{Hom}}_{\mathcal{A}^{\mathrm{t}}}(X, Y) = \underline{\mathsf{Hom}}_{\mathcal{A}}(Y, X)\,.$$
Similarly, the *transpose* of a skew $\mathcal{V}$-category is the $\mathcal{V}$-category obtained in the analogous way.

The correspondence which sends a $\mathcal{V}$-category to its transpose canonically extends to a 2-functor
$$\begin{aligned}(-)^{\mathrm{t}}\colon (\mathcal{V}\text{-}\mathcal{C}at)^{\mathrm{co}} &\to \overline{\mathcal{V}}\text{-}\mathcal{C}at\\ \mathcal{A} &\mapsto \mathcal{A}^{\mathrm{t}}\,,\end{aligned}$$
where the decoration "co" indicates that the orientation of the 2-cells of $\mathcal{V}\text{-}\mathcal{C}at$ are reversed.

2.3. — Let $\mathcal{V}$ be a monoidal category. Recall that if $\mathcal{V}$ admits limits indexed by a certain category, then so does $\mathcal{V}$-$\mathcal{C}at$. Let us spell this out in the particular case of binary products. If $\mathcal{A}$ and $\mathcal{B}$ are two $\mathcal{V}$-categories, the product $\mathcal{V}$-category $\mathcal{A}\times\mathcal{B}$ can be described in the following way:

- $\mathsf{Ob}(\mathcal{A}\times\mathcal{B}) = \mathsf{Ob}(\mathcal{A})\times\mathsf{Ob}(\mathcal{B})$,
- if X, X' are objects of $\mathcal{A}$ and Y, Y' are objects of $\mathcal{B}$, then
$$\underline{\mathsf{Hom}}_{\mathcal{A}\times\mathcal{B}}((X,Y),(X',Y')) = \underline{\mathsf{Hom}}_{\mathcal{A}}(X,X')\times\underline{\mathsf{Hom}}_{\mathcal{B}}(Y,Y')\,,$$
- if X, X', X'' are objects of $\mathcal{A}$ and Y, Y', Y'' are objects of $\mathcal{B}$, the composition of $\mathcal{A}\times\mathcal{B}$ is given by
$$\begin{array}{c} \big(\underline{\mathsf{Hom}}_{\mathcal{A}}(X',X'')\times\underline{\mathsf{Hom}}_{\mathcal{B}}(Y',Y'')\big)\otimes\big(\underline{\mathsf{Hom}}_{\mathcal{A}}(X,X')\times\underline{\mathsf{Hom}}_{\mathcal{B}}(Y,Y')\big) \\ \big\downarrow{\scriptstyle (p_1\otimes p_1,\, p_2\otimes p_2)} \\ \big(\underline{\mathsf{Hom}}_{\mathcal{A}}(X',X'')\otimes\underline{\mathsf{Hom}}_{\mathcal{A}}(X,X')\big)\times\big(\underline{\mathsf{Hom}}_{\mathcal{B}}(Y',Y'')\otimes\underline{\mathsf{Hom}}_{\mathcal{B}}(Y,Y')\big) \\ \big\downarrow{\scriptstyle \circ\times\circ} \\ \underline{\mathsf{Hom}}_{\mathcal{A}}(X,X'')\times\underline{\mathsf{Hom}}_{\mathcal{B}}(Y,Y'') \end{array}\quad,$$
where p_1 and p_2 denote the first and second projections of the cartesian product,
- if X is an object of $\mathcal{A}$ and Y is an object of $\mathcal{B}$, the identity morphism of (X,Y) is
$$I \xrightarrow{(1_X,1_Y)} \underline{\mathsf{Hom}}_{\mathcal{A}}(X,X)\times\underline{\mathsf{Hom}}_{\mathcal{B}}(Y,Y)\quad.$$

2.4. — Recall that if $F\colon\mathcal{V}\to\mathcal{V}'$ is a lax monoidal functor between monoidal categories, then F induces a 2-functor
$$F_*\colon \mathcal{V}\text{-}\mathcal{C}at\to\mathcal{V}'\text{-}\mathcal{C}at$$
sending a $\mathcal{V}$-category $\mathcal{A}$ to a $\mathcal{V}$-category $F_*(\mathcal{A})$ with same set of objects as $\mathcal{A}$ and
$$\underline{\mathsf{Hom}}_{F_*(\mathcal{A})}(X,Y) = F(\underline{\mathsf{Hom}}_{\mathcal{A}}(X,Y))\,.$$

We will call *monoidal functor* a lax monoidal functor whose structural natural transformations are isomorphisms. We will say that a functor $F\colon\mathcal{V}\to\mathcal{V}'$ between two monoidal categories is *anti-monoidal* if it is monoidal considered as a functor $F\colon\overline{\mathcal{V}}\to\mathcal{V}'$, with the notation of 2.1.

2.5. — We say that a monoidal category $\mathcal{V}$ is *closed on the right* if, for every object Y of $\mathcal{V}$, the functor $-\otimes Y$ admits a right adjoint, which will then be denoted by $\underline{\mathsf{Hom}}_r(Y,-)$. In this case, for X, Y and Z three objects of $\mathcal{V}$, we have a natural isomorphism
$$\mathsf{Hom}_{\mathcal{V}}(X\otimes Y, Z)\simeq\mathsf{Hom}_{\mathcal{V}}(X,\underline{\mathsf{Hom}}_r(Y,Z))\,.$$
If $\mathcal{V}$ is closed on the right, there is an obvious $\mathcal{V}$-category, denoted by $\mathcal{V}_r$, whose objects are the same as those of $\mathcal{V}$ and whose Hom are given by $\underline{\mathsf{Hom}}_r(X,Y)$.

Dually we say that $\mathcal{V}$ is *closed on the left* if, for every object X of $\mathcal{V}$, the functor $X \otimes -$ admits a right adjoint, which will be denoted by $\underline{\mathsf{Hom}}_l(X, -)$. We then have a natural isomorphism

$$\mathsf{Hom}_{\mathcal{V}}(X \otimes Y, Z) \simeq \mathsf{Hom}_{\mathcal{V}}(Y, \underline{\mathsf{Hom}}_l(X, Z))\,.$$

If $\mathcal{V}$ is closed on the left, there is an obvious skew $\mathcal{V}$-category, denoted by $\mathcal{V}_l$, whose objects are the same as those of $\mathcal{V}$ and whose Hom are given by $\underline{\mathsf{Hom}}_l(X, Y)$.

We say that $\mathcal{V}$ is *biclosed* if it is closed both on the left and on the right. In this case, we have a canonical isomorphism

$$\underline{\mathsf{Hom}}_l(X, \underline{\mathsf{Hom}}_r(Y, Z)) \simeq \underline{\mathsf{Hom}}_r(Y, \underline{\mathsf{Hom}}_l(Y, Z))$$

natural in X, Y and Z in $\mathcal{V}$.

2.6. — Let $\mathcal{V}$ be a monoidal category closed on the right. Suppose that $\mathcal{V}$ admits binary products. Then the binary product defines a $\mathcal{V}$-functor

$$\times \colon \mathcal{V}_r \times \mathcal{V}_r \to \mathcal{V}_r\,.$$

This $\mathcal{V}$-functor is given on objects by

$$(X, Y) \mapsto X \times Y\,,$$

and, if X, X', Y, Y' are four objects of $\mathcal{V}$, on $\underline{\mathsf{Hom}}$ by the morphism

$$\underline{\mathsf{Hom}}_r(X, X') \times \underline{\mathsf{Hom}}_r(Y, Y') \to \underline{\mathsf{Hom}}_r(X \times Y, X' \times Y')$$

obtained by adjunction from the composite

$$\begin{array}{c} \big(\underline{\mathsf{Hom}}_r(X, X') \times \underline{\mathsf{Hom}}_r(Y, Y')\big) \otimes \big(X \times Y\big) \\ \Big\downarrow {\scriptstyle (p_1 \otimes p_1,\, p_2 \otimes p_2)} \\ \big(\underline{\mathsf{Hom}}_r(X, X') \otimes X\big) \times \big(\underline{\mathsf{Hom}}_r(Y, Y') \otimes Y\big) \xrightarrow{\mathrm{ev} \times \mathrm{ev}} X' \times Y'\,, \end{array}$$

where p_1, p_2 denote the two projections of the binary product and ev the evaluation morphism of the right internal Hom.

Similarly, if $\mathcal{V}$ is closed on the left, we have a canonical skew $\mathcal{V}$-functor

$$\times \colon \mathcal{V}_l \times \mathcal{V}_l \to \mathcal{V}_l\,.$$

2.7. — Suppose now that $\mathcal{V}$ is a biclosed monoidal category and let $\mathcal{A}$ be a $\mathcal{V}$-category. A *covariant presheaf* over $\mathcal{A}$ is a $\mathcal{V}$-functor

$$F \colon \mathcal{A} \to \mathcal{V}_r\,.$$

We denote by $\underline{\mathsf{Hom}}(\mathcal{A}, \mathcal{V}_r)$ the category of covariant presheaves over $\mathcal{A}$ and of $\mathcal{V}$-natural transformations between them.

A *contravariant presheaf* over $\mathcal{A}$ is a skew $\mathcal{V}$-functor

$$F \colon \mathcal{A}^{\mathrm{t}} \to \mathcal{V}_l\,.$$

We denote by $\underline{\mathsf{Hom}}(\mathcal{A}^{\mathrm{t}}, \mathcal{V}_l)$ the category of contravariant presheaves over $\mathcal{A}$ and of skew $\mathcal{V}$-natural transformation between them.

Example 2.8. — Let a be an object of a $\mathcal{V}$-category $\mathcal{A}$. Then $\underline{\mathsf{Hom}}_{\mathcal{A}}(a,-)$ is a covariant presheaf over $\mathcal{A}$, and $\underline{\mathsf{Hom}}_{\mathcal{A}}(-,a)$ is a contravariant presheaf over $\mathcal{A}$.

As we shall now see, covariant and contravariant presheaves admit a useful description in terms of left and right modules.

2.9. — Let $\mathcal{V}$ be a biclosed monoidal category and let $\mathcal{A}$ be a $\mathcal{V}$-category. A *right $\mathcal{A}$-module* consists of

- a family $(F_a)_{a\in\mathsf{Ob}(\mathcal{A})}$ of objects of $\mathcal{V}$,
- for every objects a and a' of $\mathcal{A}$, a morphism of $\mathcal{V}$

$$\rho_{a,a'}\colon F_{a'}\otimes\underline{\mathsf{Hom}}_{\mathcal{A}}(a,a')\to F_a\,,$$

such that

- for every objects a, a' and a'' of $\mathcal{A}$, the diagram

$$\begin{array}{ccc} F_{a''}\otimes\underline{\mathsf{Hom}}_{\mathcal{A}}(a',a'')\otimes\underline{\mathsf{Hom}}_{\mathcal{A}}(a,a') & \xrightarrow{F_{a''}\otimes\circ} & F_{a''}\otimes\underline{\mathsf{Hom}}_{\mathcal{A}}(a,a'') \\ {\scriptstyle\rho_{a',a''}\otimes\underline{\mathsf{Hom}}_{\mathcal{A}}(a,a')}\downarrow & & \downarrow{\scriptstyle\rho_{a,a''}} \\ F'_a\otimes\underline{\mathsf{Hom}}_{\mathcal{A}}(a,a') & \xrightarrow[\rho_{a,a'}]{} & F_a \end{array}$$

commutes,

- for every object a of $\mathcal{A}$, the diagram

$$\begin{array}{ccc} F_a\simeq F_a\otimes I & \xrightarrow{F_A\otimes 1_a} & F_a\otimes\underline{\mathsf{Hom}}_{\mathcal{A}}(a,a) \\ & \searrow^{=} & \downarrow{\scriptstyle\rho_{a,a}} \\ & & F_a \end{array}$$

commutes.

If F and F' are two right $\mathcal{A}$-modules, a *morphism of right $\mathcal{A}$-modules* $u\colon F\to F'$ consists of a family $(u_a\colon F_a\to F'_a)_{a\in\mathsf{Ob}(\mathcal{A})}$ of morphisms of $\mathcal{V}$ such that for every objects a and a' of $\mathcal{A}$, the square

$$\begin{array}{ccc} F_{a'}\otimes\underline{\mathsf{Hom}}_{\mathcal{A}}(a,a') & \xrightarrow{\rho_{a,a'}} & F_a \\ {\scriptstyle u_{a'}\otimes\underline{\mathsf{Hom}}_{\mathcal{A}}(a,a')}\downarrow & & \downarrow{\scriptstyle u_a} \\ F'_{a'}\otimes\underline{\mathsf{Hom}}_{\mathcal{A}}(a,a') & \xrightarrow[\rho'_{a,a'}]{} & F'_a \end{array}$$

commutes. Right $\mathcal{A}$-modules and their morphisms form a category that we will denote by $\mathrm{Mod}_{\mathcal{A}}$.

The notions of *left $\mathcal{A}$-module* and of *morphism of left $\mathcal{A}$-modules* are defined analogously but with an action morphism of type

$$\lambda_{a,a'}\colon F_a\otimes\underline{\mathsf{Hom}}_{\mathcal{A}}(a,a')\to F_{a'}\,.$$

We denote the category of left $\mathcal{A}$-modules by ${}_{\mathcal{A}}\mathrm{Mod}$.

Proposition 2.10. — *Let $\mathcal{V}$ be a biclosed monoidal category and let $\mathcal{A}$ be a $\mathcal{V}$-category. We have isomorphisms of categories*

$$ {}_{\mathcal{A}}\mathrm{Mod} \xrightarrow{\sim} \underline{\mathsf{Hom}}(\mathcal{A}, \mathcal{V}_r)\,, $$

and

$$ \mathrm{Mod}_{\mathcal{A}} \xrightarrow{\sim} \underline{\mathsf{Hom}}(\mathcal{A}^{\mathrm{t}}, \mathcal{V}_l)\,. $$

Proof. — This follows from the adjunctions between the tensor product and $\underline{\mathsf{Hom}}_r$ and $\underline{\mathsf{Hom}}_l$. We leave the details to the reader. □

Remark 2.11. — In practice, in this paper, we will produce contravariant presheaves and then use the previous proposition to obtain right modules and do computations using the laws of right modules.

As we shall see in the next section, one specific property of the Gray tensor product is that its monoidal unit is the terminal object. We now develop some enriched category theory with this additional hypothesis.

2.12. — A monoidal category $(\mathcal{V}, \otimes, I)$ is said to be *with projections* if $\mathcal{V}$ admits finite products and the tensor unit of $\mathcal{V}$ is a terminal object. In this case, if X and Y are two objects of $\mathcal{V}$, we get "projections"

$$ X \xleftarrow{\ \sim\ } X \otimes I \xleftarrow{X \otimes p_X} X \otimes Y \xrightarrow{p_Y \otimes Y} I \otimes Y \xrightarrow{\ \sim\ } Y \quad , $$

where $p_Z \colon Z \to I$ denotes the unique morphism to the terminal object. In particular, we get a morphism

$$ \pi = (\pi_1, \pi_2) \colon X \otimes Y \to X \times Y\,, $$

natural in X and Y in $\mathcal{V}$.

The cartesian product defines a monoidal structure on $\mathcal{V}$ and we will denote by $\mathcal{V}^\times$ the resulting monoidal category. By default, $\mathcal{V}$ will be endowed with the monoidal product $\otimes$ but sometimes, to emphasize this, we will denote this monoidal category by $\mathcal{V}^\otimes$.

With this notation, the morphism π shows that the identify functor of $\mathcal{V}$ is a monoidal functor from $\mathcal{V}^\times$ to $\mathcal{V}^\otimes$. It is also monoidal considered with values in $\overline{\mathcal{V}^\otimes}$. In particular, we get 2-functors

$$ \mathcal{V}^\times\text{-}\mathcal{C}at \to \mathcal{V}^\otimes\text{-}\mathcal{C}at \quad \text{and} \quad \mathcal{V}^\times\text{-}\mathcal{C}at \to \overline{\mathcal{V}^\otimes}\text{-}\mathcal{C}at\,. $$

If the morphism $\pi \colon X \otimes Y \to X \times Y$ is an epimorphism for every objects X and Y of $\mathcal{V}$, we will say that $\mathcal{V}$ has *jointly surjective projections*. In this case, the two 2-functors above are injective on objects and fully faithful.

Suppose moreover that $\mathcal{V}$ is a cartesian closed and that $\mathcal{V}^\otimes$ is monoidal biclosed. We will denote by $\mathcal{V}_{\mathrm{cart}}$ the $\mathcal{V}^\times$-category of objects of $\mathcal{V}$. In particular, if X and Y are two objects of $\mathcal{V}$, then $\underline{\mathsf{Hom}}_{\mathcal{V}_{\mathrm{cart}}}(X, Y) = \underline{\mathsf{Hom}}_{\mathcal{V}}(X, Y)$, where $\underline{\mathsf{Hom}}_{\mathcal{V}}$ denotes the cartesian internal Hom of $\mathcal{V}$. Based on the above, $\mathcal{V}_{\mathrm{cart}}$ can also be considered as either a $\mathcal{V}^\otimes$-category or a $\overline{\mathcal{V}^\otimes}$-category. By the Yoneda lemma, using the morphism π, we get canonical morphisms

$$ \underline{\mathsf{Hom}}_{\mathcal{V}}(X, Y) \to \underline{\mathsf{Hom}}_r(X, Y) \quad \text{and} \quad \underline{\mathsf{Hom}}_{\mathcal{V}}(X, Y) \to \underline{\mathsf{Hom}}_l(X, Y)\,. $$

These morphisms are monomorphisms if $\mathcal{V}$ has jointly surjective projections. In any case, they induce a $\mathcal{V}^{\otimes}$-functor and a $\overline{\mathcal{V}^{\otimes}}$-functor

$$\mathcal{V}_{\text{cart}} \to \mathcal{V}_r \quad \text{and} \quad \mathcal{V}_{\text{cart}} \to \mathcal{V}_l \,.$$

2.13. — Let $\mathcal{V}$ be a monoidal category with projections. If X, Y and Z are three objects of $\mathcal{V}$, we have a canonical natural morphism

$$\varphi\colon X \otimes (Y \times Z) \to (X \otimes Y) \times Z \,,$$

given on components by

$$X \otimes (Y \times Z) \xrightarrow{X \otimes p_1} X \otimes Y \quad \text{and} \quad X \otimes (Y \times Z) \xrightarrow{\pi_2} Y \times Z \xrightarrow{p_2} Z \,,$$

where p_1 and p_2 denote the projections of the cartesian product, and π_1 and π_2 the "projections" of the tensor product. We will not need this fact but one can show that φ is a tensorial strength on the functor $- \times Z$.

Suppose moreover that $\mathcal{V}$ is cartesian closed and monoidal biclosed. Then the morphism φ induces a natural morphism

$$\lambda\colon\ \underline{\mathsf{Hom}}_l(X, \underline{\mathsf{Hom}}(Y, Z)) \to \underline{\mathsf{Hom}}(Y, \underline{\mathsf{Hom}}_l(X, Z))$$

that makes the following square commutative

$$\begin{array}{ccc} \underline{\mathsf{Hom}}_l(X, \underline{\mathsf{Hom}}(Y, Z)) & \xrightarrow{\lambda} & \underline{\mathsf{Hom}}(Y, \underline{\mathsf{Hom}}_l(X, Z)) \\ \downarrow & & \downarrow \\ \underline{\mathsf{Hom}}_l(X, \underline{\mathsf{Hom}}_r(Y, Z)) & \xrightarrow{\sim} & \underline{\mathsf{Hom}}_r(Y, \underline{\mathsf{Hom}}_l(X, Z)) \end{array} \quad ,$$

where the vertical arrows are induced by the canonical morphism from $\underline{\mathsf{Hom}}$ to $\underline{\mathsf{Hom}}_r$. Explicitly, the morphism λ is obtained by adjunction from

$$\begin{array}{l} X \otimes \big(\underline{\mathsf{Hom}}_l(X, \underline{\mathsf{Hom}}(Y, Z)) \times Y\big) \\ \qquad\qquad {\scriptstyle\varphi}\big\downarrow \\ \big(X \otimes \underline{\mathsf{Hom}}_l(X, \underline{\mathsf{Hom}}(Y, Z))\big) \times Y \xrightarrow{\text{ev} \times Y} \underline{\mathsf{Hom}}(Y, Z) \times Y \xrightarrow{\text{ev}} Z \end{array} \quad ,$$

where ev denotes the two evaluation morphisms. It follows from the commutative square above that if $\mathcal{V}$ has jointly surjective projections, then λ is a monomorphism.

Remark 2.14. — The previous paragraph can be dualized to $\underline{\mathsf{Hom}}_r$. In particular, there is a canonical natural morphism

$$\underline{\mathsf{Hom}}_r(X, \underline{\mathsf{Hom}}(Y, Z)) \to \underline{\mathsf{Hom}}(Y, \underline{\mathsf{Hom}}_r(X, Z)) \,.$$

3. Preliminaries on strict ω-categories

3.1. — For any $n \geqslant 1$, we denote by $n\text{-}\mathcal{C}at$ the category of (small) strict n-categories, that is, the category of categories enriched in $(n-1)\text{-}\mathcal{C}at$ with the cartesian monoidal structure, the category $0\text{-}\mathcal{C}at$ being the category of sets. We will drop the adjective "strict" and simply refer to "strict n-categories" as "n-categories".

We have a canonical inclusion

$$(n-1)\text{-}\mathcal{C}at \hookrightarrow n\text{-}\mathcal{C}at\,,$$

which sends an $(n-1)$-category to the n-category obtained by adding only trivial n-cells. This inclusion admits a right adjoint $\tau_{n-1}\colon n\text{-}\mathcal{C}at \to (n-1)\text{-}\mathcal{C}at$, and the category $\omega\text{-}\mathcal{C}at$ of ω-categories is obtained as the limit

$$\cdots \longrightarrow 2\text{-}\mathcal{C}at \xrightarrow{\tau_1} 1\text{-}\mathcal{C}at \xrightarrow{\tau_0} 0\text{-}\mathcal{C}at \quad .$$

For any $n \geq 0$, we have a canonical fully faithful functor

$$n\text{-}\mathcal{C}at \hookrightarrow \omega\text{-}\mathcal{C}at\,,$$

admitting both a left and a right adjoint, and whose image consists exactly of those ω-categories with only trivial k-cells (that is, identities) for $k > n$. We shall always identify this image with the category of n-categories, and consider the previous fully faithful functor as an inclusion.

Let us introduce some notation. Let C be an ω-category. For an n-cell x and $0 \leq k < n$, we denote respectively by

$$s_k(x) \text{ and } t_k(x)$$

the k-dimensional source and target of x. In the case where $k = n-1$, we also write $s(x)$ and $t(x)$. For $k > n$, we denote by

$$1^k_x$$

the k-dimensional unit on x. When $k = n+1$, we also write 1_x. Given two n-cells x and y such that $s_k(x) = t_k(x)$, with $k < n$, we denote by

$$x *_k y$$

their k-composition. More generally, for $n > k$ and $m > k$, x an n-cell, y an m-cell, such that $s_k(x) = t_k(y)$, then if $n < m$, we set

$$x *_k y := 1^m_x *_k y\,,$$

and if $n > m$, we set

$$x *_k y := x *_k 1^n_y\,.$$

For any $n \geq 0$, we define D_n to be the n-category freely generated by a unique one n-cell. Explicitly, D_n has exactly one non-trivial n-cell e_n, and exactly two non-trivial k-cells for $0 \leq k < n$ given by $s_k(e_n)$ and $t_k(e_n)$.

$$\mathrm{D}_0 = \bullet, \quad \mathrm{D}_1 = \bullet \to \bullet, \quad \mathrm{D}_2 = \bullet \overset{\Downarrow}{\rightrightarrows} \bullet\,, \quad \cdots$$

3.2. — The category $\omega\text{-}\mathcal{C}at$ has several interesting monoidal structures. First, it is cartesian closed and the associated internal Hom is denoted by

$$\underline{\mathsf{Hom}}(A, B)\,,$$

for A and B two ω-categories. The n-cells of this ω-category are in bijection with the ω-functors

$$\mathrm{D}_n \times A \to B\,.$$

In particular, the 0-cells are simply the ω-functors $A \to B$. For $n \geq 1$, the n-cells are referred to as *strict n-transformations*. Explicitly, given two ω-functors $u, v \colon A \to B$, a strict n-transformation α, with 0-source u and 0-target v, is a family (α_x) of n-cells of B, indexed by the 0-cells x of A, such that

- $s_0(\alpha_x) = u(x)$ and $t_0(\alpha_x) = v(x)$,
- for every k-cell x of C, with $k \geq 1$, we have

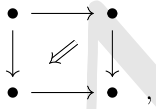
$$\alpha_{t_0(x)} *_0 u(x) = v(x) *_0 \alpha_{s_0(x)} \,.$$

The category $\omega\text{-}\mathcal{C}at$ is enriched over itself via the cartesian product, and we have a "fixed point" property: the category of categories enriched in $(\omega\text{-}\mathcal{C}at, \times, \mathrm{D}_0)$ is canonically isomorphic to $\omega\text{-}\mathcal{C}at$ itself. In particular, we have an ω-category

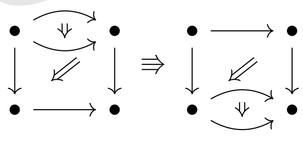
$$\omega\text{-}\mathcal{C}at_{\mathrm{cart}} \,,$$

whose 0-cells are (small) ω-categories, whose 1-cells are ω-functors and whose n-cells, with $n > 1$, are the strict $(n-1)$-transformations.

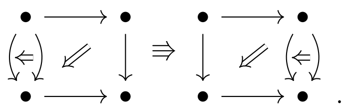
3.3. — Another fundamental monoidal category structure on $\omega\text{-}\mathcal{C}at$ comes from the so-called *(oplax) Gray tensor product* (see for example [**4**, Appendix A]), denoted by $\otimes$. To give an intuition, the tensor product $\mathrm{D}_1 \otimes \mathrm{D}_1$ is a square with a non-trivial 2-cell



whereas the cartesian product $\mathrm{D}_1 \times \mathrm{D}_1$ is a commutative square. The unit is the terminal ω-category D_0 (as for the cartesian structure) and the Gray tensor product thus defines a monoidal structure with projections in the sense of 2.12. This monoidal structure is *not* symmetrical (and not even braided). For example, we have

$\mathrm{D}_1 \otimes \mathrm{D}_2 =$ 

and

$\mathrm{D}_2 \otimes \mathrm{D}_1 =$ 

3.4. — Let A and B be two categories. If x is an i-cell of A and y a j-cell of B, then there is an associated $(i+j)$-cell $x \otimes y$ in $A \otimes B$. Explicitly, this cell corresponds to the ω-functor

$$\mathrm{D}_{i+j} \xrightarrow{\tilde{c}} \mathrm{D}_i \otimes \mathrm{D}_j \xrightarrow{\tilde{x} \otimes \tilde{y}} A \otimes B \quad ,$$

where c denotes the *principal cell* of $\mathrm{D}_i \otimes \mathrm{D}_j$, that is, its unique non-trivial $(i+j)$-cell (see for instance [**3**, paragraph B.1.5]) and $\tilde{z}$ denotes the ω-functor $\mathrm{D}_k \to C$ associated to a k-cell z of an ω-category C.

We will not need this fact in this paper but one can show that cells of the form $x \otimes y$ generate $A \otimes B$ under composition.

3.5. — The Gray tensor product is biclosed, with internal Hom denoted by $\underline{\mathsf{Hom}}_{\mathrm{oplax}}$ and $\underline{\mathsf{Hom}}_{\mathrm{lax}}$, respectively, so that we have

$$\begin{aligned}\mathsf{Hom}(A \otimes B, C) &\simeq \mathsf{Hom}(A, \underline{\mathsf{Hom}}_{\mathrm{oplax}}(B, C)) \\ &\simeq \mathsf{Hom}(B, \underline{\mathsf{Hom}}_{\mathrm{lax}}(A, C))\,,\end{aligned}$$

for A, B and C three ω-categories. Moreover, by the monoidal preliminaries, this last bijection can be promoted to a natural isomorphism

$$\underline{\mathsf{Hom}}_{\mathrm{lax}}(A, \underline{\mathsf{Hom}}_{\mathrm{oplax}}(B, C)) \simeq \underline{\mathsf{Hom}}_{\mathrm{oplax}}(B, \underline{\mathsf{Hom}}_{\mathrm{lax}}(A, C))\,.$$

The n-cells of $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A, B)$ (resp. $\underline{\mathsf{Hom}}_{\mathrm{lax}}(A, B)$) are in bijection with the ω-functors

$$\mathrm{D}_n \otimes A \to B \quad (\text{resp. } A \otimes \mathrm{D}_n \to B)\,.$$

In particular, the 0-cells of both $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A, B)$ and $\underline{\mathsf{Hom}}_{\mathrm{lax}}(A, B)$ are simply the ω-functors $A \to B$. For $n \geq 1$, an n-cell of $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A, B)$ (resp. $\underline{\mathsf{Hom}}_{\mathrm{lax}}(A, B)$) is called an *oplax n-transformation* (resp. *lax n-transformation*). For an explicit description of oplax 1-transformations (also simply referred to as *oplax transformations*), see for example [**4**, paragraph 1.9].

3.6. — By definition, if $u, v \colon A \to B$ are two ω-functors, an oplax transformation α from u to v corresponds to an ω-functor $h \colon \mathrm{D}_1 \otimes A \to B$ making the following diagram commutative

$$\begin{array}{ccc} \mathrm{D}_0 \otimes A \simeq A & & \\ {\scriptstyle \sigma\otimes A}\downarrow & \searrow^{u} & \\ \mathrm{D}_1 \otimes A & \xrightarrow{\ h\ } & A \\ {\scriptstyle \tau\otimes A}\uparrow & \nearrow_{v} & \\ \mathrm{D}_0 \otimes A \simeq A & & \end{array}\ ,$$

where $\sigma, \tau \colon \mathrm{D}_0 \to \mathrm{D}_1$ correspond to the source and the target of the non-trivial 1-cell of D_1. By adjunction, it also corresponds to an ω-functor $k \colon A \to \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_1, B)$ making the diagram

$$\begin{array}{ccc} & \nearrow^{u} & B \simeq \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_0, B) \\ & & \uparrow {\scriptstyle \underline{\mathsf{Hom}}_{\mathrm{lax}}(\sigma, A)} \\ A & \xrightarrow{\ k\ } & \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_1, B) \\ & & \downarrow {\scriptstyle \underline{\mathsf{Hom}}_{\mathrm{lax}}(\tau, A)} \\ & \searrow_{v} & B \simeq \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_0, B) \end{array}$$

commutative. This leads us to set

$$\Gamma B = \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_1, B)\,.$$

This ω-category is the *ω-category of cylinders in B*. A k-cell in this ω-category is called a *k-cylinder in B*. In other words, a k-cylinder in B is an ω-functor $\mathrm{D}_1 \otimes \mathrm{D}_k \to B$.

If $\beta\colon \mathrm{D}_1 \otimes \mathrm{D}_k \to B$ is such a k-cylinder, the image of the principal cell of $\mathrm{D}_1 \otimes \mathrm{D}_k$ (see 3.4) in B will be called the principal cell of β. We will denote it by β_k.

As a particular case of the compatibilities between $\underline{\mathsf{Hom}}_{\mathrm{lax}}$ and $\underline{\mathsf{Hom}}_{\mathrm{oplax}}$ (see 3.5), we get that if A and B are two ω-categories, then we have a natural isomorphism

$$\Gamma\,\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,B) \simeq \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,\Gamma B)\,.$$

This isomorphism will play a major role in this work.

3.7. — If C is an ω-category, then the ω-category ΓC is naturally the object of morphisms of a category internal to ω-$\mathcal{C}at$. Indeed, the functors

$$\sigma,\tau\colon \mathrm{D}_0 \to \mathrm{D}_1,\quad \kappa\colon \mathrm{D}_1 \to \mathrm{D}_0 \quad\text{and}\quad \nabla\colon \mathrm{D}_1 \to \mathrm{D}_1 \amalg_{\mathrm{D}_0} \mathrm{D}_1\,,$$

corresponding respectively to 0 and 1 in D_1, the unit of 0 in D_0 and the total composition of $\mathrm{D}_1 \amalg_{\mathrm{D}_0} \mathrm{D}_1$, define a cocategory internal to categories, and hence internal to ω-categories, and by applying the functor $\underline{\mathsf{Hom}}_{\mathrm{lax}}(-,C)$ which sends colimits to limits, we get ω-functors

$$\mathbb{s},\mathbb{t}\colon \Gamma C \to C,\quad \mathbb{k}\colon C \to \Gamma C \quad\text{and}\quad *_c\colon \Gamma C \times_C \Gamma C \to \Gamma C$$

defining a structure of category internal to ω-categories. If x is a cell of C, we will often denote $\mathbb{k}(x)$ by $\mathbb{1}_x$.

3.8. — Let C be an ω-category and let c and d be two objects of C. For every 1-cell $u\colon c' \to c$, we have an ω-functor

$$\Gamma\,\underline{\mathsf{Hom}}_C(u,d)\colon \Gamma\,\underline{\mathsf{Hom}}_C(c,d) \to \Gamma\,\underline{\mathsf{Hom}}_C(c',d)\,.$$

If α is a cell in $\Gamma\,\underline{\mathsf{Hom}}_C(c,d)$, we will denote its image by this ω-functor by $\alpha *_r u$.

Similarly, if $v\colon d \to d'$ is a 1-cell of C, we have an ω-functor

$$\Gamma\,\underline{\mathsf{Hom}}_C(c,v)\colon \Gamma\,\underline{\mathsf{Hom}}_C(c,d) \to \Gamma\,\underline{\mathsf{Hom}}_C(c,d')\,,$$

and if α is a cell in $\Gamma\,\underline{\mathsf{Hom}}_C(c,d)$, its image will be denoted by $v *_l \alpha$.

Remark 3.9. — We will come back to these operations in terms of modules in 5.3 and Remark 5.7.

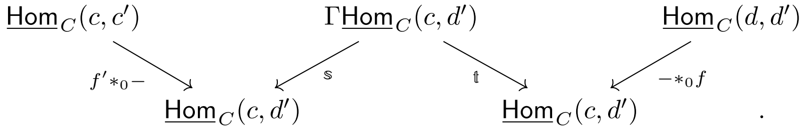
In [**5**, Section 4] (see also [**6**, Appendix A]), the authors describe the ω-category ΓC in an inductive way. We now rephrase their description.

3.10. — Let C be an ω-category. The ω-category ΓC can be described (up to isomorphism) as a category enriched in ω-categories in the following way:

- The objects of the ΓC are the 1-cells of C.
- If $f\colon c \to d$ and $f'\colon c' \to d'$ are two objects of ΓC, we have

$$\begin{aligned}&\underline{\mathsf{Hom}}_{\Gamma C}(f,f')\\ &\qquad = \underline{\mathsf{Hom}}_C(c,c') \times_{\underline{\mathsf{Hom}}_C(c,d')} \Gamma\underline{\mathsf{Hom}}_C(c,d') \times_{\underline{\mathsf{Hom}}_C(c,d')} \underline{\mathsf{Hom}}_C(d,d')\,,\end{aligned}$$

where this iterated fiber product denotes the limit of the diagram



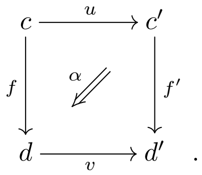
Concretely, a k-cell in this $\underline{\mathsf{Hom}}$ is a triple (u, α, v) in

$$\underline{\mathsf{Hom}}_C(c, c')_k \times \Gamma\underline{\mathsf{Hom}}_C(c, d')_k \times \underline{\mathsf{Hom}}_C(d, d')_k$$

such that

$$\mathbb{s}(\alpha) = f' *_0 u \quad \text{and} \quad \mathbb{t}(\alpha) = v *_0 f\,. \tag{$*$}$$

This formula is an ω-categorification of the formula for a 1-cylinder, i.e., a 2-square:



In other words, a $(k+1)$-cylinder in C is given by its 0-source $f\colon c \to d$ and its 0-target $f'\colon c' \to d'$, a k-cell u in C of 0-source c and 0-target c', a k-cell v in C of 0-source d and 0-target d', and a k-cylinder α in $\underline{\mathsf{Hom}}_C(c, d')$ satisfying the relations $(*)$.

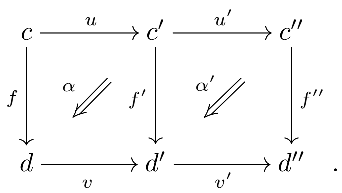
– If f, f', f'' are three objects of ΓC, the enriched composition

$$\underline{\mathsf{Hom}}_{\Gamma C}(f', f'') \times \underline{\mathsf{Hom}}_{\Gamma C}(f, f') \to \underline{\mathsf{Hom}}_{\Gamma C}(f, f'')$$

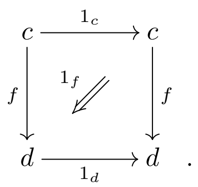
is given by

$$((u', \alpha', v'), (u, \alpha, v)) \mapsto (u' *_0 u, (v' *_l \alpha) *_c (\alpha' *_r u), v' *_0 v)\,.$$

This formula is an ω-categorification of the formula for composing the diagram



– If $f\colon c \to d$ is an object of ΓC, its unit is the 1-cylinder $(1_c, \mathbb{1}_f, 1_d)$, corresponding to the commutative square



3.11. — A *Gray ω-category* is a category enriched in $(\omega\text{-}\mathcal{C}at, \otimes, \mathrm{D}_0)$ and a *skew Gray ω-category* a skew category enriched in $(\omega\text{-}\mathcal{C}at, \otimes, \mathrm{D}_0)$.

If $\mathbb{C}$ is a Gray ω-category, its objects will also be called 0-*cells* of $\mathbb{C}$. If c and d are two 0-cells of $\mathbb{C}$, the k-cells of the ω-category $\underline{\mathsf{Hom}}_{\mathbb{C}}(c, d)$ will be called $(k+1)$-*cells* of $\mathbb{C}$. By definition, if x is such a cell, the 0-*source* $s_0(x)$ of x is c and its 0-*target* $t_0(x)$ is d. The composition of $x *_i y$ of two k-cells of $\underline{\mathsf{Hom}}_{\mathbb{C}}(c, d)$ will be denoted $x *_{i+1} y$ if the cells x and y are considered as $(k+1)$-cells of $\mathbb{C}$.

If a, b and c are three 0-cells of $\mathbb{C}$, the enriched composition will be denoted by

$$*_0 \colon \underline{\mathsf{Hom}}_{\mathbb{C}}(b, c) \otimes \underline{\mathsf{Hom}}_{\mathbb{C}}(a, b) \to \underline{\mathsf{Hom}}_{\mathbb{C}}(a, c)\,.$$

If x is a k-cell of 0-source b and 0-target c and y is an l-cell of 0-source a and 0-target b, we will denote by $x *_0 y$ the cell obtained by applying the enriched composition to the $(k+l-2)$-cell $x \otimes y$ of $\underline{\mathsf{Hom}}_{\mathbb{C}}(b, c) \otimes \underline{\mathsf{Hom}}_{\mathbb{C}}(a, b)$. This cell is a $(k+l-2)$-cell of $\underline{\mathsf{Hom}}_{\mathbb{C}}(a, c)$. In other words, $x *_0 y$ is a $(k+l-1)$-cell of $\mathbb{C}$. Its 0-source is a and its 0-target c. Note that the enriched composition is uniquely determined by all the $x *_0 y$.

of $\mathbb{C}$ is a k-cell. The composition $x *_y y$ of two 2-cells of $\mathbb{C}$ is a 3-cell. In general, the exchange rule in $\mathbb{C}$ for the cells x and y do not hold on the nose but up to the oriented 3-cell $x *_0 y$:

$$(x *_0 t(y)) *_1 (s(x) *_0 y) \overset{x *_0 y}{\Longrightarrow} (t(x) *_0 y) *_1 (\beta *_0 s(y))$$

(see [**3**, Proposition B.1.14]). We refer the reader to [**3**, Section B.1] for more details on the elementary structure of Gray ω-categories.

Similar definitions and notation apply to skew Gray ω-categories.

3.12. — Enriched functors between Gray ω-categories will be called *Gray ω-functors*. Similarly, enriched functors between skew Gray ω-categories will be called *skew Gray ω-functors*. A Gray or skew Gray ω-functor is uniquely determined by its action on cells. More precisely, morphisms between Gray or skew Gray ω-functor can be described as functions on k-cells for every $k \geqslant 0$ that are compatible with sources, targets, compositions $*_i$ and units.

Enriched natural transformations between Gray or skew Gray ω-functors will be called *strict transformations*. Explicitly, if $F, G \colon \mathbb{C} \to \mathbb{C}'$ are two Gray ω-functors (or two skew Gray ω-functors), a strict transformation $\alpha \colon F \Rightarrow G$ consists of the data of a 1-cell

$$\alpha_c \colon F(c) \to G(c)$$

for every 0-cell c of $\mathbb{C}$, such that, for every k-cell x, with $k \geqslant 1$, one has

$$\alpha_{t_0(x)} *_0 F(x) = G(x) *_0 \alpha_{s_0(x)}\,.$$

3.13. — We will denote by

$$\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \quad (\text{resp. } \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}})$$

the Gray ω-category (resp. the skew Gray ω-category) whose objects are ω-categories and whose Hom are

$$\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A, B) \quad (\text{resp. } \underline{\mathsf{Hom}}_{\mathrm{lax}}(A, B))\,,$$

where A and B are two ω-categories. By definition, the 0-cells of $\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$ (resp. of $\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}$) are ω-categories, its 1-cells are ω-functors, its 2-cells are oplax (resp. lax) transformations and, for $n > 2$, its n-cells are oplax (resp. lax) $(n-1)$-transformations.

3.14. — Since the monoidal unit of the Gray tensor product is the terminal ω-category, by 2.12, for every ω-categories A and B, we get an ω-functor

$$\pi = (\pi_1, \pi_2)\colon A \otimes B \to A \times B\,.$$

Proposition 3.15. — *If A and B are two ω-categories, the ω-functor*

$$\pi\colon A \otimes B \to A \times B$$

is an epimorphism.

Proof. — The ω-category $A \times B$ is generated under composition by cells of the form (x, b), where x is a cell of A and b a 0-cell of B, and of the form (a, y), where a is a 0-cell of A and y a cell of B. It thus suffices to show that these cells are in the image of the ω-functor π. But if b is an object of B, considering the commutative diagram

$$\begin{array}{ccc} A \otimes \mathrm{D}_0 & \xrightarrow{A \otimes b} & A \otimes B \\ {\scriptstyle\wr}\big\downarrow{\scriptstyle\pi} & & \big\downarrow{\scriptstyle\pi} \\ A \times \mathrm{D}_0 & \xrightarrow[A \times b]{} & A \times B \end{array} \quad ,$$

we get that for every cell x of A, the cell (x, b) is in the image of $\pi\colon A \otimes B \to A \times B$. A similar argument shows that cells of the form (a, y) are in the image of π, thereby proving the result. $\square$

Remark 3.16. — We will not need it but one can actually prove that the ω-functor π of the proposition is surjective on cells. More generally, if x is a k-cell of A and y is a l-cell of B, then we have $\pi(x \otimes y) = (1_x, 1_y)$, where 1_x and 1_y denote the iterated units of x and y in dimension $k + l$.

3.17. — In the language of 2.12, the previous proposition states that the monoidal category $(\omega\text{-}\mathcal{C}at, \otimes, \mathrm{D}_0)$ has jointly surjective projections. We can thus apply the considerations of 2.12 and 2.13. Let us start by 2.12.

We get that the category of ω-categories embeds both in the category of Gray ω-categories and in the category of skew Gray ω-categories. Moreover, we have canonical monomorphisms

$$\underline{\mathsf{Hom}}(A, B) \hookrightarrow \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A, B) \quad \text{and} \quad \underline{\mathsf{Hom}}(A, B) \hookrightarrow \underline{\mathsf{Hom}}_{\mathrm{lax}}(A, B)\,,$$

which we will treat as inclusions.

We thus have canonical Gray and skew Gray ω-functors

$$\omega\text{-}\mathcal{C}at_{\mathrm{cart}} \hookrightarrow \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \quad \text{and} \quad \omega\text{-}\mathcal{C}at_{\mathrm{cart}} \hookrightarrow \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}\,.$$

3.18. — Let us now apply 2.13 to $(\omega\text{-}\mathcal{C}at, \otimes, \mathrm{D}_0)$. If D, A, B are three ω-categories, we have a canonical monomorphism

$$\lambda\colon \underline{\mathsf{Hom}}_{\mathrm{lax}}(D, \underline{\mathsf{Hom}}(A, B)) \hookrightarrow \underline{\mathsf{Hom}}(A, \underline{\mathsf{Hom}}_{\mathrm{lax}}(D, B))$$

making the square

$$\begin{array}{ccc} \underline{\mathsf{Hom}}_{\mathrm{lax}}(D, \underline{\mathsf{Hom}}(A, B)) & \xhookrightarrow{\lambda} & \underline{\mathsf{Hom}}(A, \underline{\mathsf{Hom}}_{\mathrm{lax}}(D, B)) \\ \downarrow & & \downarrow \\ \underline{\mathsf{Hom}}_{\mathrm{lax}}(D, \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A, B)) & \xrightarrow{\sim} & \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A, \underline{\mathsf{Hom}}_{\mathrm{lax}}(D, B)) \end{array} ,$$

commute. We will treat λ as an inclusion. We thus get a factorization

$$\underline{\mathsf{Hom}}_{\mathrm{lax}}(D, \underline{\mathsf{Hom}}(A, B)) \hookrightarrow \underline{\mathsf{Hom}}(A, \underline{\mathsf{Hom}}_{\mathrm{lax}}(D, B)) \hookrightarrow \underline{\mathsf{Hom}}_{\mathrm{lax}}(D, \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A, B))$$

of the canonical inclusion. This applies in particular in the case where $D = \mathrm{D}_1$ in which we get inclusions

$$\Gamma\, \underline{\mathsf{Hom}}(A, B) \hookrightarrow \underline{\mathsf{Hom}}(A, \Gamma B) \hookrightarrow \Gamma\, \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A, B)\,.$$

We end the section by some considerations on the dualities of $\omega\text{-}\mathcal{C}at$.

3.19. — If $S \subset \mathbb{N}^*$ is a subset of the set of positive integers, then we will denote by

$$D_S\colon \omega\text{-}\mathcal{C}at \to \omega\text{-}\mathcal{C}at$$

the ω-functor sending an ω-category C to the ω-category obtained from C by reversing the orientation of all the cells whose dimension belongs to S. It is immediate that D_S is an involutive endofunctor of $\omega\text{-}\mathcal{C}at$. Actually, up to isomorphism, all the autoequivalences of $\omega\text{-}\mathcal{C}at$ are of the form D_S. We will sometimes refer to these autoequivalences as *dualities*.

Several special cases play an important role in the theory of $\omega\text{-}\mathcal{C}at$:

- If $S = \mathbb{N}^*$, then $D_{\mathbb{N}^*}$ is denoted by $D_\circ$ and is called the *total dual*. We simply write C° for the total dual of an ω-category C.
- If $S = 2\mathbb{N}+1$ is the set of odd integers, then $D_{2\mathbb{N}+1}$ is denoted by D_{op} and is called the *odd dual*. We simply write C^{op} for the odd dual of an ω-category C.
- If $S = 2\mathbb{N}^*$ is the set of positive even integers, then $D_{2\mathbb{N}^*}$ is denoted by D_{co} and is called the *even dual*. We simply write C^{co} for the odd dual of an ω-category C.
- If $S = \{1\}$, then $D_{\{1\}}$ is denoted by D_{t} and is called the *transpose*. We simply write C^{t} for the transpose of C. This coincide with the transpose of C in the sense of 2.2 when C is considered as a category enriched over $\omega\text{-}\mathcal{C}at$ endowed with the cartesian product.

By composing all these special dualities, we get a group of eight dualities. In particular, if C is an ω-category, we get seven other ω-categories

$$C^\circ,\quad C^{\mathrm{op}},\quad C^{\mathrm{co}},\quad C^{\mathrm{t}},\quad C^{\mathrm{to}} = (C^\circ)^{\mathrm{t}},\quad C^{\mathrm{top}} = (C^{\mathrm{op}})^{\mathrm{t}},\quad C^{\mathrm{cot}} = (C^{\mathrm{co}})^{\mathrm{t}}\,.$$

Note that this group of dualities is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^3$, a natural basis (as a $\mathbb{Z}/2\mathbb{Z}$-module) being given by D_{op}, D_{co} and D_{t}.

We are now going to recall the compatibilities of the dualities of $\omega\text{-}\mathcal{C}at$ with the Gray tensor product.

Proposition 3.20. — *Let A and B be two ω-categories. There are canonical isomorphisms*

$$(A \otimes B)^{\mathrm{op}} \simeq B^{\mathrm{op}} \otimes A^{\mathrm{op}}, \quad (A \otimes B)^{\mathrm{co}} \simeq B^{\mathrm{co}} \otimes A^{\mathrm{co}}, \quad (A \otimes B)^{\circ} \simeq A^{\circ} \otimes B^{\circ}\,,$$

natural in A and B. In other words, the functors

$$D_{\mathrm{co}}, D_{\mathrm{op}} \colon \omega\text{-}\mathcal{C}at \to \omega\text{-}\mathcal{C}at$$

are anti-monoidal and the functor

$$D_{\circ} \colon \omega\text{-}\mathcal{C}at \to \omega\text{-}\mathcal{C}at$$

is monoidal, ω-$\mathcal{C}at$ being endowed with the Gray tensor product.

Proof. — See for instance [**4**, Proposition A.22]. □

Remark 3.21. — Besides the trivial duality, D_{op}, D_{co} and $D_{\circ}$ are the only dualities of ω-$\mathcal{C}at$ that are either monoidal or anti-monoidal.

Remark 3.22. — It follows from the previous proposition that if A and B are two ω-categories, then we have natural isomorphisms

$$\begin{aligned} \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,B)^{\mathrm{op}} &\simeq \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A^{\mathrm{op}},B^{\mathrm{op}})\,,\\ \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,B)^{\mathrm{co}} &\simeq \underline{\mathsf{Hom}}_{\mathrm{lax}}(A^{\mathrm{co}},B^{\mathrm{co}})\,,\\ \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,B)^{\circ} &\simeq \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A^{\circ},B^{\circ})\,. \end{aligned}$$

3.23. — Let $\mathbb{C}$ be a Gray ω-category. By the previous proposition, from $\mathbb{C}$, we can get two skew Gray ω-categories $(D_{\mathrm{op}})_*(\mathbb{C})$ and $(D_{\mathrm{op}})_*(\mathbb{C})$, and a Gray ω-category $(D_{\circ})_*(\mathbb{C})$, obtained by applying these dualities $\underline{\mathsf{Hom}}$-wise. Note that in these new Gray or skew Gray ω-categories, the 1-cells are never reversed and in some sense there is a shift by 1 of the dimensions that are reversed. With this in mind, we set

$$\mathbb{C}^{\mathrm{op}} = \big((D_{\mathrm{co}})_*(\mathbb{C})\big)^{\mathrm{t}}, \quad \mathbb{C}^{\mathrm{co}} = (D_{\mathrm{op}})_*(\mathbb{C}), \quad \mathbb{C}^{\circ} = \big((D_{\circ})_*(\mathbb{C})\big)^{\mathrm{t}}\,.$$

Then $\mathbb{C}^{\mathrm{op}}$ is a Gray ω-category, and $\mathbb{C}^{\mathrm{co}}$ and $\mathbb{C}^{\circ}$ are skew Gray ω-categories. We also set

$$\mathbb{C}^{\mathrm{top}} = (\mathbb{C}^{\mathrm{op}})^{\mathrm{t}}, \quad \mathbb{C}^{\mathrm{cot}} = (\mathbb{C}^{\mathrm{co}})^{\mathrm{t}}, \quad \mathbb{C}^{\mathrm{to}} = (\mathbb{C}^{\circ})^{\mathrm{t}}\,.$$

To sum up, from a Gray ω-category $\mathbb{C}$, we get three other Gray ω-categories

$$\mathbb{C}^{\mathrm{op}}, \quad \mathbb{C}^{\mathrm{cot}}, \quad \mathbb{C}^{\mathrm{to}}$$

and four skew Gray ω-categories

$$\mathbb{C}^{\mathrm{t}}, \quad \mathbb{C}^{\mathrm{top}}, \quad \mathbb{C}^{\mathrm{co}}, \quad \mathbb{C}^{\circ}\,,$$

and no other duality of ω-$\mathcal{C}at$ produces a Gray or a skew Gray ω-category.

3.24. — Using the previous paragraph, one can interpret the dualities D_{co}, D_{op} and $D_{\circ}$ of ω-categories as Gray ω-functors. More precisely, one can check using 3.22 that these dualities induce isomorphisms of Gray ω-categories

$$\begin{aligned} D_{\mathrm{co}} &\colon (\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}})^{\mathrm{top}} \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\,,\\ D_{\mathrm{op}} &\colon (\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}})^{\mathrm{co}} \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\,,\\ D_{\circ} &\colon (\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}})^{\mathrm{to}} \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\,. \end{aligned}$$

4. Comma ω-categories

4.1. — Consider a diagram

$$A \xrightarrow{f} C \xleftarrow{g} B$$

in ω-$\mathcal{C}at$. The *oplax comma ω-category* $A \downarrow_C B$, also denoted by $f \downarrow g$, is the universal ω-category endowed with 2-square

$$\begin{array}{ccccc} & & A \downarrow_C B & & \\ & {}^{p_1}\swarrow & & \searrow^{p_2} & \\ A & & \overset{\gamma}{\Longrightarrow} & & B \\ & {}_{f}\searrow & & \swarrow_{g} & \\ & & C & & \end{array}$$

in ω-$\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$, that is, where p_1 and p_2 are ω-functors and κ is an *oplax* transformation. This means that if T is an ω-category endowed with a similar 2-square

$$\begin{array}{ccccc} & & T & & \\ & {}^{a}\swarrow & & \searrow^{b} & \\ A & & \overset{\lambda}{\Longrightarrow} & & B \\ & {}_{f}\searrow & & \swarrow_{g} & \\ & & C & & \end{array},$$

then there exists a unique ω-functor $h_{a,\lambda,b} \colon T \to A$ that factors the diagram in the sense that

$$p_1 h_{a,\lambda,b} = a, \quad p_2 h_{a,\lambda,b} = b \quad \text{and} \quad \kappa *_0 h_{a,\lambda,b} = \lambda\,.$$

By 3.6, and with its notation, it is immediate that the data of such a 2-square is equivalent to the data of an ω-functor from T to the limit of

$$\begin{array}{ccccccc} A & & & \Gamma C & & & B \\ & {}_{f}\searrow & {}_{s}\swarrow & & \searrow_{t} & \swarrow_{g} & \\ & C & & & & C & \end{array}.$$

This shows that we have

$$A \downarrow_C B = A \times_C \Gamma C \times_C B\,.$$

The canonical projections p_1 and p_2 are the obvious projections on A and B, and the 2-cell $\gamma \colon f p_1 \Rightarrow g p_2$ corresponds to the projection

$$\gamma \colon A \downarrow_C B \to \Gamma C\,.$$

Example 4.2. — The slice ω-categories are particular cases of comma ω-categories. Indeed, if C is an ω-category and c is an object of C, then the comma construction $c \downarrow C$ of the diagram

$$\mathrm{D}_0 \xrightarrow{c} C \xleftarrow{1_C} C$$

is canonically isomorphic to the slice ω-category ${}_{c\backslash}C$ described in [**4**, Chapter 9] (see [**3**, Proposition 7.1] for a proof). More generally if $v\colon B \to C$ is an ω-functor, then we have

$${}_{c\backslash}B = c \downarrow v\,,$$

where ${}_{c\backslash}B$ is the relative slice defined by the pullback

$$\begin{array}{ccc} {}_{c\backslash}B & \longrightarrow & {}_{c\backslash}C \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle U} \\ B & \xrightarrow[v]{} & C \end{array}\quad,$$

with U denoting the forgetful ω-functor.

Similarly, we have

$$C/_{c} = C \downarrow c$$

and, more generally, if $u\colon A \to C$ is an ω-functor,

$$A/_{c} = u \downarrow c\,.$$

4.3. — The comma construction $A \downarrow_C B$ is functorial in A and B. Indeed, if

$$\begin{array}{ccccc} A & & & & B \\ & \overset{f}{\searrow} & & \overset{g}{\swarrow} & \\ {\scriptstyle u}\big\downarrow & \overset{\alpha}{\Nearrow} & C & \overset{\beta}{\Searrow} & \big\downarrow{\scriptstyle v} \\ & \underset{f'}{\nearrow} & & \underset{g'}{\nwarrow} & \\ A' & & & & B' \end{array}$$

is a diagram in $\omega\text{-}\mathbb{C}\mathsf{at}_{\text{oplax}}$, then we get an ω-functor

$$(f,\alpha)\downarrow(\beta,g)\colon A \downarrow_C B \to A' \downarrow_C B'$$

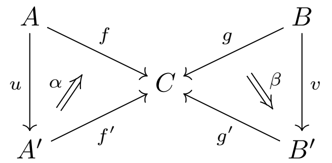
by applying the universal property of $A' \downarrow_C B'$ to the 2-square obtained by composing the diagram

$$\begin{array}{ccccc} & & A \downarrow_C B & & \\ & \overset{p_1}{\swarrow} & & \overset{p_2}{\searrow} & \\ A & & \overset{\gamma}{\Longrightarrow} & & B \\ & \overset{f}{\searrow} & & \overset{g}{\swarrow} & \\ {\scriptstyle u}\big\downarrow & \overset{\alpha}{\Nearrow} & C & \overset{\beta}{\Searrow} & \big\downarrow{\scriptstyle v} \\ & \underset{f'}{\nearrow} & & \underset{g'}{\nwarrow} & \\ A' & & & & B' \end{array}\quad.$$

Therefore, the comma construction defines a functor from the obvious category whose objects are the diagrams

$$A \xrightarrow{f} C \xleftarrow{g} B$$

and whose morphisms are diagrams



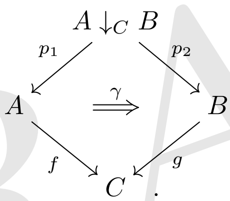
in $\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$ to the category $\omega\text{-}\mathcal{C}at$. Better, the first-named author and Maltsiniotis proved that the comma construction can be promoted to a sesquifunctor (see [**3**, Theorem B.2.6]).

The main goal of this paper is to express and prove the full functorialities of the comma construction, with respect to the higher structure of the Gray ω-category $\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$. To do so, one ingredient will be the ω-categorical universal property of the comma construction that we will now describe.

4.4. — Let

$$A \xrightarrow{f} C \xleftarrow{g} B$$

as before and consider the universal 2-square



For every ω-category T, we have an induced canonical ω-functor

$$\begin{gathered}\underline{\mathsf{Hom}}_{\mathrm{oplax}}(T, A \downarrow_C B)\\ \big\downarrow \\ \underline{\mathsf{Hom}}_{\mathrm{oplax}}(T,A) \times_{\underline{\mathsf{Hom}}_{\mathrm{oplax}}(T,C)} \underline{\mathsf{Hom}}_{\mathrm{oplax}}(T,\Gamma C) \times_{\underline{\mathsf{Hom}}_{\mathrm{oplax}}(T,C)} \underline{\mathsf{Hom}}_{\mathrm{oplax}}(T,B),\end{gathered}$$

induced by the projections p_1, γ and p_2, which we can identify with an ω-functor

$$\underline{\mathsf{Hom}}_{\mathrm{oplax}}(T, A \downarrow_C B) \longrightarrow \underline{\mathsf{Hom}}_{\mathrm{oplax}}(T,A) \underset{\underline{\mathsf{Hom}}_{\mathrm{oplax}}(T,C)}{\downarrow} \underline{\mathsf{Hom}}_{\mathrm{oplax}}(T,B)$$

using the canonical isomorphism $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(T,\Gamma C) \simeq \Gamma\, \underline{\mathsf{Hom}}_{\mathrm{oplax}}(T,C)$ of 3.6. Since the functor $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(T,-)\colon \omega\text{-}\mathcal{C}at \to \omega\text{-}\mathcal{C}at$ is a right adjoint, it commutes with fibred products and we get the following proposition:

Proposition 4.5 **(Higher universal property of the comma construction)**
If $A \xrightarrow{f} C \xleftarrow{g} B$ *is a diagram in* $\omega\text{-}\mathcal{C}at$*, then for any* ω*-category* T *the canonical morphism*

$$\underline{\mathsf{Hom}}_{\mathrm{oplax}}(T, A \downarrow_C B) \xrightarrow{\sim} \underline{\mathsf{Hom}}_{\mathrm{oplax}}(T,A) \underset{\underline{\mathsf{Hom}}_{\mathrm{oplax}}(T,C)}{\downarrow} \underline{\mathsf{Hom}}_{\mathrm{oplax}}(T,B)\,,$$

is an isomorphism.

The enriched description of the ω-category of cylinders (see 3.10) leads to a analogous description for the comma ω-category:

4.6. — If $A \xrightarrow{f} C \xleftarrow{g} B$ is a diagram in $\omega\text{-}\mathcal{C}at$, then the ω-category $A \downarrow_C B$ can be described (up to isomorphism) as a category enriched in ω-categories in the following way:

- The objects of $A \downarrow_C B$ are triples $(a, l\colon fa \to gb, b)$, where a is an object of A, b an object of B and l a 1-cell of C.
- If $(a, l\colon fa \to gb, b)$ and $(a', l'\colon fa' \to gb', b')$ are two objects of $A \downarrow_C B$, then

  $$\begin{aligned}&\underline{\mathsf{Hom}}_{A\downarrow_C B}((a,l,b),(a',l',b'))\\ &\quad = \underline{\mathsf{Hom}}_A(a,a') \times_{\underline{\mathsf{Hom}}_C(fa,gb')} \Gamma\underline{\mathsf{Hom}}_C(fa,gb') \times_{\underline{\mathsf{Hom}}_C(fa,gb')} \underline{\mathsf{Hom}}_B(b,b')\,,\end{aligned}$$

  where this iterated fiber product denotes the limit of the diagram

  $$\begin{array}{ccccccc} \underline{\mathsf{Hom}}_A(a,a') & & & \Gamma\underline{\mathsf{Hom}}_C(fa,gb') & & & \underline{\mathsf{Hom}}_B(b,b') \\ & \searrow^{l'*_0 f(-)} & \swarrow_{\mathbb{s}} & & \searrow^{\mathbb{t}} & \swarrow_{g(-)*_0 l} & \\ & \underline{\mathsf{Hom}}_C(fa,gb') & & & & \underline{\mathsf{Hom}}_C(fa,gb') & \end{array}.$$

  This ω-category is actually itself a comma ω-category, namely

  $$\underline{\mathsf{Hom}}_A(a,a') \downarrow_{\underline{\mathsf{Hom}}_C(fa,gb')} \underline{\mathsf{Hom}}_B(b,b')\,.$$

  Concretely, a k-cell in this $\underline{\mathsf{Hom}}$ is a triple (u,α,v) in

  $$\underline{\mathsf{Hom}}_A(a,a')_k \times \Gamma\underline{\mathsf{Hom}}_C(fa,gb')_k \times \underline{\mathsf{Hom}}_B(b,b')_k$$

  such that

  $$\mathbb{s}(\alpha) = l' *_0 f(u) \quad \text{and} \quad \mathbb{t}(\alpha) = g(v) *_0 l\,.$$
- If $(a,l,b), (a',l',b'), (a'',l'',b'')$ are three objects of $A \downarrow_C B$, the enriched composition

  $$\begin{array}{c}\underline{\mathsf{Hom}}_{\Gamma C}((a',l',b'),(a'',l'',b'')) \times \underline{\mathsf{Hom}}_{\Gamma C}((a,l,b),(a',l',b')) \\ \downarrow \\ \underline{\mathsf{Hom}}_{\Gamma C}((a,l,b),(a'',l'',b''))\end{array}$$

  is given by

  $$((u',\alpha',v'),(u,\alpha,v)) \mapsto (u' *_0 u, (g(v') *_l \alpha) *_c (\alpha' *_r f(u)), v' *_0 v)\,.$$
- If (a,l,b) is an object of $A \downarrow_C B$, its unit is the triple $(1_a, \mathbb{1}_f, 1_b)$.

4.7. — The comma construction we studied in this section is the *oplax* comma construction. Similarly, one can define a *lax comma construction* by replacing the oplax transformation in the 2-square of the universal property of the oplax comma construction by a lax transformation. If

$$A \xrightarrow{\ f\ } C \xleftarrow{\ g\ } B$$

is a diagram in $\omega\text{-}\mathcal{C}at$, we will denote by $u \downarrow' v$, or $A \downarrow'_C B$, the lax comma construction of u and v. Explicitly, we have

$$A \downarrow'_C B = A \times_C \Gamma' C \times_C B\,,$$

where $\Gamma'(C) = \underline{\mathsf{Hom}}_{\mathrm{oplax}}(\mathrm{D}_1, C)$.

The lax comma construction can also be defined by duality from the oplax version. Indeed, there are natural isomorphisms

$$(A \downarrow_C B)^{\mathrm{op}} \simeq B^{\mathrm{op}} \downarrow_{C^{\mathrm{op}}} A^{\mathrm{op}} \quad \text{and} \quad (A \downarrow'_C B)^{\mathrm{co}} \simeq A^{\mathrm{co}} \downarrow'_{C^{\mathrm{co}}} B^{\mathrm{co}}\,.$$

In particular, there is a natural isomorphism

$$(A \downarrow_C B)^{\circ} \simeq B^{\circ} \downarrow_{C^{\circ}} A^{\circ} \quad \text{and} \quad (A \downarrow'_C B)^{\circ} \simeq B^{\circ} \downarrow'_{C^{\circ}} A^{\circ}\,.$$

In this text, we will mainly deal with with the oplax version of the comma construction construction and therefore drop the adjective “oplax”.

5. Slices of Gray ω-categories

The purpose of this section is to define, for $\mathbb{C}$ a Gray ω-category and c an object of $\mathbb{C}$, a slice Gray ω-category $\mathbb{C}_{/c}$.

The description of the comma construction of 4.6 gives in particular an enriched description for the slices of ω-categories. We will see that this description still makes sense for Gray ω-categories. The definition will involve the ω-category

$$\Gamma(\underline{\mathsf{Hom}}_{\mathbb{C}}(d, c)) = \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_1, \underline{\mathsf{Hom}}_{\mathbb{C}}(d, c))\,,$$

where d is another object of $\mathbb{C}$, and we start the section by an analysis of the structure of this ω-category.

5.1. — The mapping

$$C \mapsto \Gamma C = \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_1, C)$$

is the object part of a skew Gray ω-functor

$$\Gamma = \underline{\mathsf{Hom}}_{\mathrm{lax}}(\mathrm{D}_1, -)\colon \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}} \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}\,.$$

5.2. — Let $\mathbb{C}$ be a Gray ω-category. For every object c of $\mathbb{C}$, by composing the two skew Gray ω-functors

$$\mathbb{C}^{\mathrm{t}} \xrightarrow{\underline{\mathsf{Hom}}_{\mathbb{C}}(-,c)} \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}} \xrightarrow{\Gamma} \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}\,,$$

we get a skew Gray ω-functor

$$\Gamma(\underline{\mathsf{Hom}}_{\mathbb{C}}(-, c))\colon \mathbb{C}^{\mathrm{t}} \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}\,.$$

By Proposition 2.10, this means that $\Gamma(\underline{\mathsf{Hom}}_{\mathbb{C}}(-, c))$ is a right $\mathbb{C}$-module in the sense of 2.9.

5.3. — Let $\mathbb{C}$ be a Gray ω-category and let a, b and c be three objects of $\mathbb{C}$. The structure of right $\mathbb{C}$-module of the previous paragraph defines an ω-functor

$$*_r\colon \Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c) \otimes \underline{\mathsf{Hom}}_{\mathbb{C}}(a,b) \to \Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(a,c)\,.$$

Moreover, the axioms of modules give that, if a, b, c and d are four objects of $\mathbb{C}$, then the diagrams

$$\begin{array}{ccc}
\Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(c,d) \otimes \underline{\mathsf{Hom}}_{\mathbb{C}}(b,c) \otimes \underline{\mathsf{Hom}}_{\mathbb{C}}(a,b) & \xrightarrow{\Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(c,d)\otimes *_0} & \Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(c,d) \otimes \underline{\mathsf{Hom}}_{\mathbb{C}}(a,c) \\
{\scriptstyle *_r\otimes\underline{\mathsf{Hom}}_{\mathbb{C}}(a,b)}\downarrow & & \downarrow{\scriptstyle *_r} \\
\Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(b,d) \otimes \underline{\mathsf{Hom}}_{\mathbb{C}}(a,b) & \xrightarrow[*_r]{} & \Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(a,d)
\end{array}$$

and

$$\begin{array}{ccc}
\Gamma\,\underline{\mathsf{Hom}}_{\mathbb{C}}(a,b) & \xrightarrow{\Gamma\,\underline{\mathsf{Hom}}_{\mathbb{C}}(a,b)\otimes 1_a} & \Gamma\,\underline{\mathsf{Hom}}_{\mathbb{C}}(a,b) \otimes \underline{\mathsf{Hom}}_{\mathbb{C}}(a,a) \\
& \searrow_{=} & \downarrow{\scriptstyle *_r} \\
& & \Gamma\,\underline{\mathsf{Hom}}_{\mathbb{C}}(a,b)
\end{array}$$

commute.

5.4. — We saw in 3.7 that, if C is an ω-category, then we have ω-functors

$$\mathbb{s},\mathbb{t}\colon \Gamma C \to C, \quad \mathbb{k}\colon C \to \Gamma C \quad \text{and} \quad *_c\colon \Gamma C \times_C \Gamma C \to \Gamma C$$

defining a structure of category internal to ω-categories. All the operations of this structure are natural in C. This means that this structure of internal category to ω-$\mathcal{C}at$ extends to a structure of internal category to the category of skew Gray ω-functors from ω-$\mathbb{C}\mathsf{at}_{\mathrm{lax}}$ to itself and strict transformations between them. More precisely, the skew Gray ω-functor $\Gamma\colon \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}} \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}$ is the object of morphisms of a category internal to the category of skew Gray ω-functors from ω-$\mathbb{C}\mathsf{at}_{\mathrm{lax}}$ to itself, the object of objects being the identity functor.

5.5. — Let $\mathbb{C}$ be a Gray ω-category and let c be an object of $\mathbb{C}$. By precomposing by the skew Gray ω-functor

$$\underline{\mathsf{Hom}}_{\mathbb{C}}(-,c)\colon \mathbb{C}^{\mathrm{t}} \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}$$

the internal category of the previous paragraph, we get that

$$\Gamma(\underline{\mathsf{Hom}}_{\mathbb{C}}(-,c))\colon \mathbb{C}^{\mathrm{t}} \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}$$

is the object of morphisms of a category internal to the category of skew Gray ω-functors from $\mathbb{C}^{\mathrm{t}}$ to ω-$\mathbb{C}\mathsf{at}_{\mathrm{lax}}$, with object of objects $\underline{\mathsf{Hom}}_{\mathbb{C}}(-,c)$ and structure maps

$$\mathbb{s},\mathbb{t}\colon \Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(-,c) \Rightarrow \underline{\mathsf{Hom}}_{\mathbb{C}}(-,c), \quad \mathbb{k}\colon \underline{\mathsf{Hom}}_{\mathbb{C}}(-,c) \Rightarrow \Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(-,c)$$

and

$$*_c\colon \Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(-,c) \times_{\underline{\mathsf{Hom}}_{\mathbb{C}}(-,c)} \Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(-,c) \Rightarrow \Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(-,c)\,.$$

By Proposition 2.10, this means that $\Gamma(\underline{\mathsf{Hom}}_{\mathbb{C}}(-,c))$ is the object of morphisms of a category internal to the category of right $\mathbb{C}$-modules and the four above structure maps correspond to morphisms of right $\mathbb{C}$-modules.

5.6. — Let $\mathbb{C}$ be a Gray ω-category and let a, b, c three objects of $\mathbb{C}$. The fact that the structure maps $\mathbb{s}$, $\mathbb{t}$, $\mathbb{k}$ and $*_c$ of the previous paragraph correspond to morphisms of right $\mathbb{C}$-modules precisely means that the squares

$$\begin{array}{ccc}
\Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c)\otimes\underline{\mathsf{Hom}}_{\mathbb{C}}(a,b) & \xrightarrow{*_r} & \Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(a,c) \\
{\scriptstyle \mathbb{e}\otimes\underline{\mathsf{Hom}}_{\mathbb{C}}(a,b)}\downarrow & & \downarrow{\scriptstyle \mathbb{e}} \\
\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c)\otimes\underline{\mathsf{Hom}}_{\mathbb{C}}(a,b) & \xrightarrow[*_0]{} & \underline{\mathsf{Hom}}_{\mathbb{C}}(a,c)
\end{array}\quad ,$$

for $\mathbb{e}$ being $\mathbb{s}$ or $\mathbb{t}$,

$$\begin{array}{ccc}
\Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c)\otimes\underline{\mathsf{Hom}}_{\mathbb{C}}(a,b) & \xrightarrow{*_r} & \Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(a,c) \\
{\scriptstyle \mathbb{k}\otimes\underline{\mathsf{Hom}}_{\mathbb{C}}(a,b)}\uparrow & & \uparrow{\scriptstyle \mathbb{k}} \\
\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c)\otimes\underline{\mathsf{Hom}}_{\mathbb{C}}(a,b) & \xrightarrow[*_0]{} & \underline{\mathsf{Hom}}_{\mathbb{C}}(a,c)
\end{array}$$

and

$$\begin{array}{ccc}
\big(\Gamma\,\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c)\times_{\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c)}\Gamma\,\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c)\big)\otimes\underline{\mathsf{Hom}}_{\mathbb{C}}(a,b) & \xrightarrow{*_c\otimes 1} & \Gamma\,\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c)\otimes\underline{\mathsf{Hom}}_{\mathbb{C}}(a,b) \\
\downarrow & & \downarrow{\scriptstyle *_r} \\
\Gamma\,\underline{\mathsf{Hom}}_{\mathbb{C}}(a,c)\times_{\underline{\mathsf{Hom}}_{\mathbb{C}}(a,c)}\Gamma\,\underline{\mathsf{Hom}}_{\mathbb{C}}(a,c) & \xrightarrow[*_c]{} & \Gamma\,\underline{\mathsf{Hom}}_{\mathbb{C}}(a,c)
\end{array}\quad ,$$

where the left vertical arrow is the composite

$$\begin{array}{c}
\big(\Gamma\,\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c)\times_{\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c)}\Gamma\,\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c)\big)\otimes\underline{\mathsf{Hom}}_{\mathbb{C}}(a,b) \\
{\scriptstyle \mathrm{can}}\downarrow \\
\big(\Gamma\,\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c)\otimes\underline{\mathsf{Hom}}_{\mathbb{C}}(a,b)\big)\times_{\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c)\otimes\underline{\mathsf{Hom}}_{\mathbb{C}}(a,b)}\big(\Gamma\,\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c)\otimes\underline{\mathsf{Hom}}_{\mathbb{C}}(a,b)\big) \\
{\scriptstyle *_r\times_{*_0}*_r}\downarrow \\
\Gamma\,\underline{\mathsf{Hom}}_{\mathbb{C}}(a,c)\times_{\underline{\mathsf{Hom}}_{\mathbb{C}}(a,c)}\Gamma\,\underline{\mathsf{Hom}}_{\mathbb{C}}(a,c)
\end{array}\quad ,$$

commute.

Remark 5.7. — Note that if a, b and c are three objects of a Gray ω-category $\mathbb{C}$, there is no natural ω-functor

$$\underline{\mathsf{Hom}}_{\mathbb{C}}(b,c)\otimes\Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(a,b)\to\Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(a,c)\,,$$

and in particular, there is no natural structure of left $\mathbb{C}$-module on $\Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(a,-)$. What is true is that $\Gamma'\underline{\mathsf{Hom}}_{\mathbb{C}}(a,-)$, where $\Gamma'(C)=\underline{\mathrm{Hom}}_{\mathrm{oplax}}(\mathrm{D}_1,C)$, is naturally a left $\mathbb{C}$-module

We can now give the definition of the slice Gray ω-categories.

5.8. — Let $\mathbb{C}$ be a Gray ω-category and let c be an object $\mathbb{C}$. We define the *slice Gray ω-category* $\mathbb{C}/_c$ in the following way:

– The objects of $\mathbb{C}/_c$ are pairs $(d, f\colon d \to c)$, where d is an object of $\mathbb{C}$ and f a 1-cell.

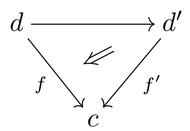
– If $(d, f\colon d \to c)$ and $(d', f'\colon d' \to c)$ are two objects of $\mathbb{C}$, we set

$$\begin{aligned}\underline{\mathsf{Hom}}_{\mathbb{C}/_c}((d,f),(d',f')) &= \underline{\mathsf{Hom}}_{\mathbb{C}}(d,d') \times_{\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)} \Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c) \times_{\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)} \{f\} \\ &= \underline{\mathsf{Hom}}_{\mathbb{C}}(d,d') \downarrow_{\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)} \{f\}\,.\end{aligned}$$

By definition, a k-cell in this $\underline{\mathsf{Hom}}$ consists of a pair (u, α), with u is a k-cell of $\underline{\mathsf{Hom}}_{\mathbb{C}}(d,d')$ and α a k-cell of $\Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)$ such that

$$\mathbb{s}(\alpha) = f' *_0 u \quad \text{and} \quad \mathbb{t}(\alpha) = 1_f\,.$$

In particular, an object of this $\underline{\mathsf{Hom}}$ corresponds to a 2-triangle



We will denote U and γ the projections

$$\begin{aligned}U\colon \underline{\mathsf{Hom}}_{\mathbb{C}/_c}((d,f),(d',f')) &\to \underline{\mathsf{Hom}}_{\mathbb{C}}(d,d') \\ \gamma\colon \underline{\mathsf{Hom}}_{\mathbb{C}/_c}((d,f),(d',f')) &\to \Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)\end{aligned}$$

so that

$$U(u,\alpha) = u \quad \text{and} \quad \gamma(u,\alpha) = \alpha\,.$$

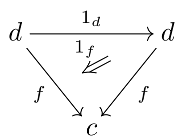
– If $(d, f\colon d \to c)$ is an object of $\mathbb{C}/_c$, the associated unit

$$\mathrm{D}_0 \to \underline{\mathsf{Hom}}_{\mathbb{C}/_c}((d,f),(d,f))$$

is given by the pair

$$\mathrm{D}_0 \xrightarrow{1_d} \underline{\mathsf{Hom}}_{\mathbb{C}}(d,d) \quad \text{and} \quad \mathrm{D}_0 \xrightarrow{f} \underline{\mathsf{Hom}}_{\mathbb{C}}(d,c) \xrightarrow{\mathbb{k}} \Gamma(\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)).$$

Concretely, it corresponds to the 2-triangle



In symbols, we have

$$1_{(d,f)} = (1_d, \mathbb{1}_f)$$

(remember that we denote $k(f)$ by $\mathbb{1}_f$).

– Let (d,f), (d,f') and (d,f'') be three objects of $\mathbb{C}/_c$. We now define the composition ω-functor

$$\underline{\mathsf{Hom}}_{\mathbb{C}/_c}((d',f'),(d'',f'')) \otimes \underline{\mathsf{Hom}}_{\mathbb{C}/_c}((d,f),(d',f')) \to \underline{\mathsf{Hom}}_{\mathbb{C}/_c}((d,f),(d'',f''))\,.$$

To define such an ω-functor we need to define two ω-functors

$$\underline{\mathsf{Hom}}_{\mathbb{C}_{/c}}((d',f'),(d'',f'')) \otimes \underline{\mathsf{Hom}}_{\mathbb{C}_{/c}}((d,f),(d',f')) \to \underline{\mathsf{Hom}}_{\mathbb{C}}(d,d'')$$
$$\underline{\mathsf{Hom}}_{\mathbb{C}_{/c}}((d',f'),(d'',f'')) \otimes \underline{\mathsf{Hom}}_{\mathbb{C}_{/c}}((d,f),(d',f')) \to \Gamma(\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c))$$

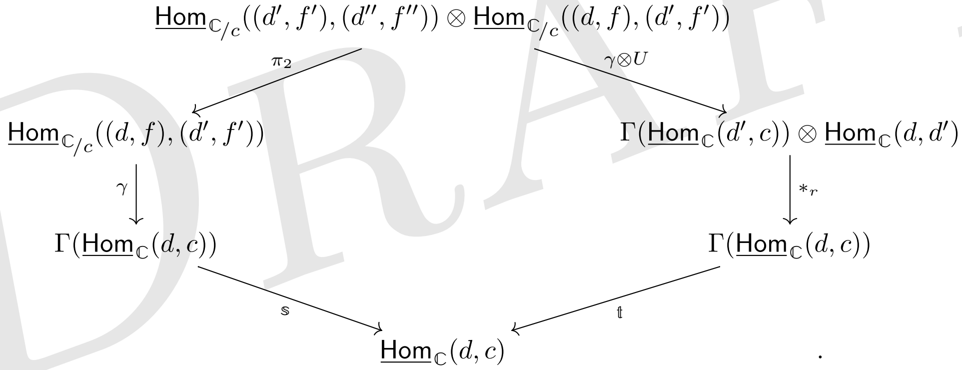
compatible with the pullback defining $\underline{\mathsf{Hom}}_{\mathbb{C}_{/c}}((d,f),(d'',f''))$. The first one is defined by composing

$$\begin{array}{c}\underline{\mathsf{Hom}}_{\mathbb{C}_{/c}}((d',f'),(d'',f'')) \otimes \underline{\mathsf{Hom}}_{\mathbb{C}_{/c}}((d,f),(d',f')) \\ \Big\downarrow{\scriptstyle U\otimes U} \\ \underline{\mathsf{Hom}}_{\mathbb{C}}(d',d'') \otimes \underline{\mathsf{Hom}}_{\mathbb{C}}(d,d') \xrightarrow{\ *_0\ } \underline{\mathsf{Hom}}_{\mathbb{C}}(d,d'')\end{array} .$$

The second is defined by composing

$$\begin{array}{c}\underline{\mathsf{Hom}}_{\mathbb{C}_{/c}}((d',f'),(d'',f'')) \otimes \underline{\mathsf{Hom}}_{\mathbb{C}_{/c}}((d,f),(d',f')) \\ \Big\downarrow \\ \Gamma(\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)) \times_{\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)} \Gamma(\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)) \xrightarrow{\ *_c\ } \Gamma(\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c))\end{array} ,$$

where the vertical morphism is induced by the following commutative hexagon

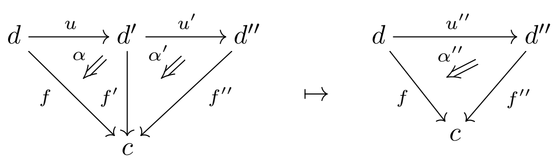


.

The fact that this indeed defines a composition ω-functor will be proven below. In symbols, we have

$$(u',\alpha') *_0 (u,\alpha) = \big(u' *_0 u, \alpha *_c (\alpha' *_r u)\big) .$$

Note that this is an ω-categorification of the composition of triangles



,

with

$$u'' = u'u \quad \text{and} \quad \alpha'' = \alpha *_1 (\alpha' *_0 u) .$$

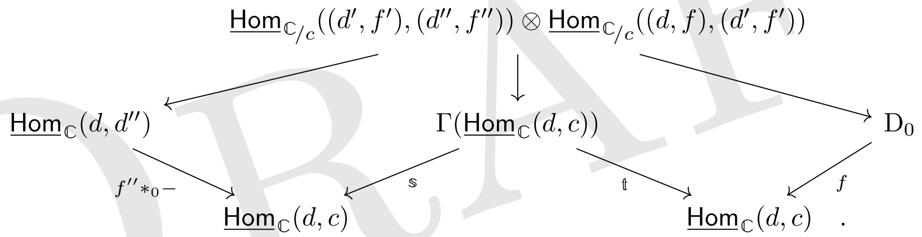
Theorem 5.9. — *If $\mathbb{C}$ is a Gray ω-category and c is an object $\mathbb{C}$, then $\mathbb{C}_{/c}$ as described above is indeed a Gray ω-category.*

Proof. — We start by proving that the composition of $\mathbb{C}_{/c}$ described in the previous paragraph is well defined. We first have to prove that the hexagon announced to be commutative is indeed commutative. Fix (u, α) a k-cell of $\underline{\mathsf{Hom}}_{\mathbb{C}_{/c}}((d, f), (d', f'))$ and (u', α') a k-cell of $\underline{\mathsf{Hom}}_{\mathbb{C}_{/c}}((d', f'), (d'', f''))$. If we evaluate the left part of the hexagon on $(u', \alpha') \otimes (u, \alpha)$, we get $\mathbb{s}(\alpha)$. Evaluating the right part, we get $\mathbb{t}(\alpha' *_r u)$. But

$$\mathbb{t}(\alpha' *_r u) = \mathbb{t}(\alpha') *_0 u = f' *_0 u = \mathbb{s}(\alpha)\,,$$

where the first equality follows from the fact that t is a morphism of right $\mathbb{C}$-modules (see the first square of 5.6 for $\mathbb{e} = \mathbb{t}$). A priori, we have shown that the hexagon is commutative on "pure tensors" $(u', \alpha') \otimes (u, \alpha)$. Nevertheless, since each of the equalities we are using comes from commutative diagrams, this algebraic proof can be transformed into a diagrammatic proof showing that the hexagon is commutative, without any restriction(1). From now on, we will freely use this technique to show commutativity of diagrams starting from a tensor product.

To prove that the composition of $\mathbb{C}_{/c}$ is well defined, we now have to show the commutativity of the diagram



For the left square of the diagram, we have

$$\mathbb{s}(\alpha *_c (\alpha' *_r u)) = \mathbb{s}(\alpha' *_r u) = \mathbb{s}(\alpha') *_0 u = (f'' *_0 u') *_0 u = f'' *_0 (u' *_0 u)\,,$$

using first the internal category structure of the ω-category of cylinders (see 3.7), then the fact that $\mathbb{s}$ is a morphism of right $\mathbb{C}$-modules (see the first commutative square of 5.6 for $\mathbb{e} = \mathbb{s}$) and finally the associativity of the Gray ω-category $\mathbb{C}$. This proves that the left square commutes. As for the right square, we have

$$\mathbb{t}(\alpha *_c (\alpha' *_r u)) = \mathbb{t}(\alpha) = 1_f\,,$$

using again the internal category structure of the ω-category of cylinders (see 3.7). This ends the proof that the composition of $\mathbb{C}_{/c}$ is well defined.

We now have to check the axioms of Gray ω-categories. Let us first prove the associativity. Fix a k-cell (u'', α'') of $\underline{\mathsf{Hom}}_{\mathbb{C}_{/c}}((d'', f''), (d''', f'''))$. We have to prove

(1) A less elegant argument to conclude that the diagram is commutative is that the "pure tensors" $a \otimes b$ in a Gray tensor product $A \otimes B$ form a generating set.

that

$$\big(u'',\alpha''\big) *_0 \big((u',\alpha') *_0 (u,\alpha)\big) = \Big(u'' *_0 (u' *_0 u),\ \big(\alpha *_c (\alpha' *_r u)\big) *_c \big(\alpha'' *_r (u' *_0 u)\big)\Big)$$

equals

$$\big((u'',\alpha'') *_0 (u',\alpha')\big) *_0 (u,\alpha) = \Big((u'' *_0 u') *_0 u,\ \alpha *_c \big[\big(\alpha' *_c (\alpha'' *_r u')\big) *_r u\big]\Big)\,.$$

The equality of the first components follows from the associativity of the composition of $\mathbb{C}$. As for the second components, we have

$$\begin{aligned}\alpha *_c \big[\big(\alpha' *_c (\alpha'' *_r u')\big) *_r u\big] &= \alpha *_c \big[\big(\alpha' *_r u\big) *_c \big((\alpha'' *_r u') *_r u\big)\big]\\ &= \alpha *_c \big[\big(\alpha' *_r u\big) *_c \big(\alpha'' *_r (u' *_0 u)\big)\big]\\ &= \big(\alpha *_c (\alpha' *_r u)\big) *_c \big(\alpha'' *_r (u' *_0 u)\big)\,,\end{aligned}$$

where the first equality follows from the fact that $*_c$ is a morphism of right $\mathbb{C}$-modules (see the last commutative square of 5.6), the second from the fact that $*_r$ is a right action (see the commutative square of 5.3) and the last from the associativity of the operation $*_c$ (see 3.7). This ends the proof that the composition of $\mathbb{C}/_c$ is associative.

Finally, we prove the axioms involving units. For the left unit axiom, we have

$$(u,\alpha) *_0 1_{(d,f)} = (u,\alpha) *_0 (1_d, \mathbb{1}_f) = (u *_0 1_d, \mathbb{1}_f *_c (\alpha *_r 1_d)) = (u,\alpha)\,,$$

where the last equality uses the axiom of units in $\mathbb{C}$, the structure of category of 3.7 and the fact that $*_r$ is a right action (see the commutative triangle of 5.3). Finally, for the right unit axiom, we have

$$1_{(d',f')} *_0 (u,\alpha) = (1_{d'}, \mathbb{1}_{f'}) *_0 (u,\alpha) = (1_{d'} *_0 u, \alpha *_c (\mathbb{1}_{f'} *_r u)) = (u,\alpha)\,,$$

where the last equality uses the axiom of units in $\mathbb{C}$, the fact that $\mathbb{1}$ is a morphism of right $\mathbb{C}$-modules (see the second commutative square of 5.6) and the structure of internal category of 3.7. □

Remark 5.10. — The existence of slice Gray ω-categories was first conjectured by the first-named author and Maltsiniotis [**4**, conjecture C.24].

5.11. — Let $\mathbb{C}$ be a Gray ω-category et let c be an object of $\mathbb{C}$. We have a canonical Gray ω-functor

$$U\colon \mathbb{C}/_c \to \mathbb{C}\,,$$

called the *forgetful Gray ω-functor*. It is defined on objects by

$$(d,f) \mapsto d\,,$$

and, if $(d,f),(d',f')$ are two objects, on morphisms by the projection

$$U\colon\ \underline{\mathsf{Hom}}_{\mathbb{C}/_c}((d,f),(d',f')) \to \underline{\mathsf{Hom}}_{\mathbb{C}}(d,d')$$

(see 5.8).

Proposition 5.12. — *Let C be an ω-category and let c be an object of C. Then, we have a canonical natural isomorphism*

$$\iota(C)/_c \simeq \iota(C/_c)$$

commuting to forgetful morphisms, where ι denotes the inclusion functor from ω-categories to Gray ω-categories.

Proof. — This is true by design of slice Gray ω-categories, and more precisely, by the description of the slice ω-categories that follows from the description of comma ω-categories given in 4.6. $\square$

Proposition 5.13. — *Let $\mathbb{C}$ be a Gray ω-category and let c and d be two objects of $\mathbb{C}$. Then the fiber of the forgetful Gray ω-functor $\mathbb{C}_{/c} \to \mathbb{C}$ at d is canonically isomorphic to $\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)^\circ$.*

Proof. — Denote by U_d this fiber. By definition, its objects are 1-cells $d \to c$ of $\mathbb{C}$, and if f, f' are two such objects, we have

$$\underline{\mathsf{Hom}}_{U_d}(f,f') = \{f'\} \times_{\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)} \Gamma\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c) \times_{\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)} \{f\} = \{f'\} \downarrow_{\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)} \{f\}\,.$$

This means that a k-cell of this ω-category is a k-cylinder α in $\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)$ such that $\mathbb{s}(\alpha) = 1_{f'}$ and $\mathbb{t}(\alpha) = 1_f$. Moreover, the enriched composition simplifies to

$$\begin{array}{ccc} \underline{\mathsf{Hom}}_{U_d}(f',f'') \otimes \underline{\mathsf{Hom}}_{U_d}(f,f') & \to & \underline{\mathsf{Hom}}_{U_d}(f,f'') \\ (\alpha',\alpha) & \mapsto & \alpha' *_c \alpha \end{array}$$

(as $\alpha' *_c (\alpha *_r 1_d) = \alpha' *_c \alpha$).

But in general, if a and a' are two objects of an ω-category A, we have a canonical isomorphism

$$\{a\} \downarrow_A \{a'\} \xrightarrow{\sim} \underline{\mathsf{Hom}}_A(a,a')^\circ$$

(see [**1**, Proposition B.6.2]), sending a k-cylinder α in A to its principal cell α_k (see 3.6). We thus have

$$\underline{\mathsf{Hom}}_{U_d}(f,f') \simeq \underline{\mathsf{Hom}}_{\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)}(f',f)^\circ = \underline{\mathsf{Hom}}_{\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)^\circ}(f,f')\,,$$

this isomorphism sending a k-cylinder α in $\underline{\mathsf{Hom}}_{\mathbb{C}}(c,d)$ to its principal cell α_{k+1}. Now if (α',α) is in $\underline{\mathsf{Hom}}_{U_d}(f',f'')_k \times \underline{\mathsf{Hom}}_{U_d}(f,f')_k$, then

$$(\alpha' *_c \alpha)_{k+1} = \alpha'_{k+1} *_0 \alpha_{k+1}\,.$$

This shows that the isomorphism

$$\underline{\mathsf{Hom}}_{U_d}(f,f') \simeq \underline{\mathsf{Hom}}_{\underline{\mathsf{Hom}}_{\mathbb{C}}(d,c)^\circ}(f,f')$$

is compatible with compositions, thereby proving the result. $\square$

5.14. — Let C be a strict ω-category and let c be an object of C. If D is any duality of ω-$\mathcal{C}at$, then by "conjugating" the slice construction $C_{/c}$ by D, one gets another slice construction. If D does not reverse the 1-cells, we set

$$C\overset{\mathrm{D}}{/}c = D\big(D(C)_{/c}\big)\,.$$

In the case where D reverses the 1-cells, this notation would be misleading as $D\big(D(C)_{/c}\big)$ is actually a slice below c (and not above c). In particular, if D is the total dual, following the notation of [**4**], we set

$$_{c\backslash}C = (C^\circ{}_{/c})^\circ\,.$$

Now if D is a general duality reversing the 1-cells, then denoting by D' the unique duality such that $D' \circ D$ is the total dual (in particular, D' does not reverse the 1-cells), one gets

$$D\big(D(C)/_{c}\big) = D'\Big(\big(D'(C)^{\circ}/_{c}\big)^{\circ}\Big) = D'\big({}_{c}\backslash D'(C)\big)\,.$$

Therefore, if D reverses the 1-cells, we set

$$\overset{\mathrm{D}'}{c\backslash} C = D\big(D(C)/_{c}\big)\,.$$

This means that we only decorate (over or under) slices by dualities that do not reverse the 1-cells.

Let us now apply this to Gray ω-categories. If $\mathbb{C}$ is a Gray ω-category and c is an object of $\mathbb{C}$, we can conjugate our Gray slice construction by the three non-trivial dualities of Gray ω-categories (see 3.23) and we set

$$\mathbb{C}\overset{\mathrm{to}}{/}c = \big(\mathbb{C}^{\mathrm{to}}/c\big)^{\mathrm{to}}\,, \qquad c\overset{\mathrm{co}}{\backslash}\mathbb{C} = \big(\mathbb{C}^{\mathrm{op}}/c\big)^{\mathrm{op}}\,, \qquad c\overset{\mathrm{top}}{\backslash}\mathbb{C} = \big(\mathbb{C}^{\mathrm{cot}}/c\big)^{\mathrm{cot}}\,.$$

Note that each of these Gray ω-categories admits a forgetful Gray ω-functor to $\mathbb{C}$.

Similarly, if $\mathbb{C}$ is a skew Gray ω-category and c is an object of $\mathbb{C}$, we set

$$c\backslash\mathbb{C} = \big(\mathbb{C}^{\circ}/c\big)^{\circ}\,, \qquad c\overset{\mathrm{to}}{\backslash}\mathbb{C} = \big(c\backslash\mathbb{C}^{\mathrm{to}}\big)^{\mathrm{to}}\,, \qquad \mathbb{C}\overset{\mathrm{co}}{/}c = \big(c\backslash\mathbb{C}^{\mathrm{op}}\big)^{\mathrm{op}}\,, \qquad \mathbb{C}\overset{\mathrm{top}}{/}c = \big(c\backslash\mathbb{C}^{\mathrm{cot}}\big)^{\mathrm{cot}}\,.$$

Each of these skew Gray ω-categories admits a forgetful skew Gray ω-functor to $\mathbb{C}$.

6. Lax functorialities of the comma construction

The purpose of this section is to extend the comma construction

$$A \longrightarrow C \longleftarrow B \qquad \mapsto \qquad A \downarrow_C B$$

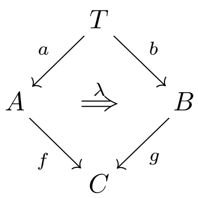
to a Gray ω-functor

$$-\downarrow_C - \colon \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C \times \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\overset{\mathrm{to}}{/}C \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\,.$$

6.1. — Let us fix ω-functors

$$A \xrightarrow{f} C \xleftarrow{g} B \qquad \text{and} \qquad A' \xrightarrow{f'} C \xleftarrow{g'} B'\,.$$

To any 2-square



in ω-$\mathbb{C}\mathsf{at}_{\text{oplax}}$, we are going to associate an ω-functor

$$K\colon \underline{\mathsf{Hom}}_{\omega\text{-}\mathbb{C}\mathsf{at}_{\text{oplax}/C}}((A,f),(A',f')) \times \underline{\mathsf{Hom}}_{\omega\text{-}\mathbb{C}\mathsf{at}^{\text{to}}_{\text{oplax}/C}}((B,g),(B',g'))$$
$$\Big\downarrow$$
$$\underline{\mathsf{Hom}}_{\text{oplax}}(T,A') \underset{\underline{\mathsf{Hom}}_{\text{oplax}}(T,C)}{\downarrow} \underline{\mathsf{Hom}}_{\text{oplax}}(T,B') \quad .$$

Recall that

$$\begin{aligned}&\underline{\mathsf{Hom}}_{\omega\text{-}\mathbb{C}\mathsf{at}_{\text{oplax}/C}}((A,f),(A',f'))\\ &\qquad = \underline{\mathsf{Hom}}_{\text{oplax}}(A,A') \times_{\underline{\mathsf{Hom}}_{\text{oplax}}(A,C)} \Gamma\, \underline{\mathsf{Hom}}_{\text{oplax}}(A,C) \times_{\underline{\mathsf{Hom}}_{\text{oplax}}(A,C)} \{f'\}\end{aligned}$$

and

$$\begin{aligned}&\underline{\mathsf{Hom}}_{\omega\text{-}\mathbb{C}\mathsf{at}^{\text{to}}_{\text{oplax}/C}}((B,g),(B',g'))\\ &\qquad = \{g\} \times_{\underline{\mathsf{Hom}}_{\text{oplax}}(B,C)} \Gamma\, \underline{\mathsf{Hom}}_{\text{oplax}}(B,C) \times_{\underline{\mathsf{Hom}}_{\text{oplax}}(B,C)} \underline{\mathsf{Hom}}_{\text{oplax}}(B,B')\,,\end{aligned}$$

so that an i-cell in the product of these two ω-categories consists of a 4-uple

$$(u,\alpha,\beta,v)$$

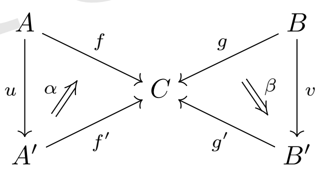
in

$$\underline{\mathsf{Hom}}_{\text{oplax}}(A,A')_i \times (\Gamma\, \underline{\mathsf{Hom}}_{\text{oplax}}(A,C))_i \times (\Gamma\, \underline{\mathsf{Hom}}_{\text{oplax}}(B,C))_i \times \underline{\mathsf{Hom}}_{\text{oplax}}(B,B')_i$$

satisfying

$$\mathbb{s}(\alpha) = f' *_0 u, \quad \mathbb{t}(\alpha) = f, \quad \mathbb{s}(\beta) = g, \quad \mathbb{t}(\beta) = g' *_0 v\,.$$

In particular, for $i=0$, we get a diagram



.

Similarly, an i-cell

$$\underline{\mathsf{Hom}}_{\text{oplax}}(T,A') \underset{\underline{\mathsf{Hom}}_{\text{oplax}}(T,C)}{\downarrow} \underline{\mathsf{Hom}}_{\text{oplax}}(T,B')$$

consists of a triple

$$(a',\delta,b')$$

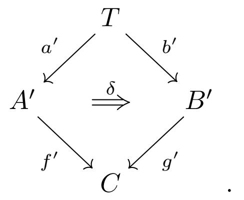
in

$$\underline{\mathsf{Hom}}_{\text{oplax}}(T,A')_i \times (\Gamma\underline{\mathsf{Hom}}_{\text{oplax}}(T,C))_i \times \underline{\mathsf{Hom}}_{\text{oplax}}(T,B')_i$$

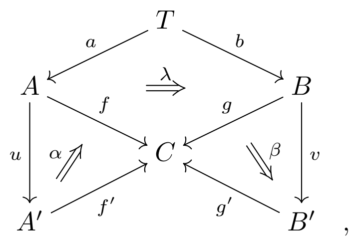
such that

$$s(\delta) = f' *_0 a' \quad \text{and} \quad t(\delta) = g' *_0 b'\,.$$

For $i = 0$, we get a 2-square



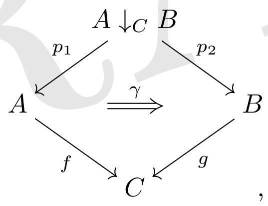
The ω-functor K is defined by categorification of the formula defining the total composite of the 2-diagram



that is, by the formula

$$(u, \alpha, \beta, v) \mapsto (u *_0 a, (\beta *_r b) *_c \lambda *_c (\alpha *_r a), v *_0 b)\,.$$

6.2. — In particular, if we apply the construction of the previous paragraph to the universal 2-square



we get an ω-functor

$$- \downarrow_C - \colon \underline{\mathsf{Hom}}_{\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}/C}}((A, f), (A', f')) \times \underline{\mathsf{Hom}}_{\omega\text{-}\mathbb{C}\mathsf{at}^{\mathrm{to}}_{\mathrm{oplax}/C}}((B, g), (B', g'))$$
$$\big\downarrow$$
$$\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A \downarrow_C B, A') \underset{\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A \downarrow_C B, C)}{\downarrow} \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A \downarrow_C B, B')$$
$$\simeq$$
$$\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A \downarrow_C B, A' \downarrow_C B') \quad .$$

It is given by the formula

$$(u, \alpha, \beta, v) \mapsto (u *_0 p_1, (\beta *_r p_2) *_c \gamma *_c (\alpha *_r p_1), v *_0 p_2)\,.$$

In particular, if $g \colon B \to C$ is an ω-functor, we have

$$\begin{aligned}(- \downarrow_C B)(u, \alpha) &= (- \downarrow_C -)(u, \alpha, 1_v, \mathbb{1}_B) \\ &= (u *_0 p_1, (1_v *_r p_2) *_c \gamma *_c (\alpha *_c p_1), \mathbb{1}_B *_0 p_2) \\ &= (u *_0 p_1, \gamma *_c (\alpha *_r p_1), p_2)\,,\end{aligned}$$

and, similarly, if $f \colon A \to C$ is an ω-functor, we have

$$(A \downarrow_C -)(\beta, v) = (p_1, (\beta *_r p_2) *_c \gamma, v *_0 p_2)\,.$$

Theorem 6.3. — *Let C be an ω-category. The comma construction extends, via the construction of the previous paragraph, to a Gray ω-functor*

$$- \downarrow_C - \colon \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C \times \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \overset{\mathrm{to}}{/} C \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\,.$$

Proof. — In this proof, to make our formula slightly more compact, we set

$$\underline{\mathsf{Hom}}_{\mathrm{ol}} := \underline{\mathsf{Hom}}_{\mathrm{oplax}}, \quad \underline{\mathsf{Hom}}_{/C} := \underline{\mathsf{Hom}}_{\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C} \quad \text{and} \quad \underline{\mathsf{Hom}}_{\overset{\mathrm{to}}{/C}} := \underline{\mathsf{Hom}}_{\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\overset{\mathrm{to}}{/}C}\,.$$

For the same reason, if $(T, T \to C)$ is an ω-category over C, we will denote it simply by T.

Let us now prove the proposition. We have to check the compatibility with the unit and the compatibility with the composition. To do so, we will use the same technique as in Theorem 5.9.

For the unit, we have

$$\begin{aligned}(- \downarrow_C -)\big(1_{(A,f)}, 1_{(B,g)}\big) &= (- \downarrow_C -)(\mathbb{1}_A, 1_f, 1_g, \mathbb{1}_B) \\ &= (\mathbb{1}_A *_0 p_1, (1_g *_r p_2) *_0 \gamma *_0 (1_f *_r p_1), \mathbb{1}_B *_0 p_2) \\ &= (p_1, \gamma, p_2) \\ &= 1_{A \downarrow_C B}\,.\end{aligned}$$

Let us now check the compatibility with composition. Consider A, A', A'', B, B', B'' six ω-categories over C. We have to prove that the two canonical ω-functors

$$\begin{gathered}\big(\underline{\mathsf{Hom}}_{/C}(A', A'') \times \underline{\mathsf{Hom}}_{\overset{\mathrm{to}}{/C}}(B', B'')\big) \otimes \big(\underline{\mathsf{Hom}}_{/C}(A, A') \times \underline{\mathsf{Hom}}_{\overset{\mathrm{to}}{/C}}(B, B')\big) \\ \Big\downarrow \\ \underline{\mathsf{Hom}}_{\mathrm{ol}}(A \downarrow_C B, A'' \downarrow_C B'') \quad ,\end{gathered}$$

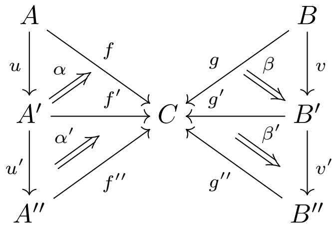
which we will describe below, are equal. Consider

$$(u', \alpha', \beta', u) \quad \text{and} \quad (u, \alpha, \beta, v)$$

cells of

$$\underline{\mathsf{Hom}}_{/C}(A', A'') \times \underline{\mathsf{Hom}}_{\overset{\mathrm{to}}{/C}}(B', B'') \quad \text{and} \quad \underline{\mathsf{Hom}}_{/C}(A, A') \times \underline{\mathsf{Hom}}_{\overset{\mathrm{to}}{/C}}(B, B')$$

respectively. When these cells are 0-cells, we get a diagram



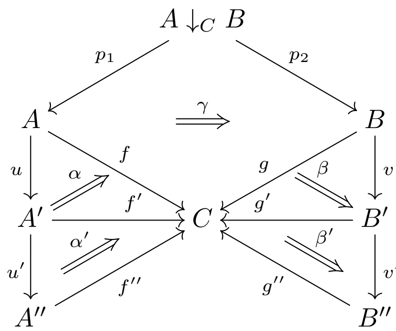
Consider first the ω-functor M defined as the composite

$$\begin{array}{c}
\big(\underline{\mathsf{Hom}}_{/C}(A', A'') \times \underline{\mathsf{Hom}}^{\mathrm{to}}_{/C}(B', B'')\big) \otimes \big(\underline{\mathsf{Hom}}_{/C}(A, A') \times \underline{\mathsf{Hom}}^{\mathrm{to}}_{/C}(B, B')\big) \\
\Big\downarrow{\scriptstyle \mathrm{can}} \\
\big(\underline{\mathsf{Hom}}_{/C}(A', A'') \otimes \underline{\mathsf{Hom}}_{/C}(A, A')\big) \times \big(\underline{\mathsf{Hom}}^{\mathrm{to}}_{/C}(B', B'') \otimes \underline{\mathsf{Hom}}_{/C}(B, B')\big) \\
\Big\downarrow{\scriptstyle *_0 \otimes *_0} \\
\underline{\mathsf{Hom}}_{/C}(A, A'') \times \underline{\mathsf{Hom}}^{\mathrm{to}}_{/C}(B, B'') \\
\Big\downarrow{\scriptstyle - \downarrow_C -} \\
\underline{\mathsf{Hom}}_{\mathrm{ol}}(A \downarrow_C B, A'' \downarrow_C B'')\,.
\end{array}$$

We have

$$\begin{aligned}
&M\big((u', \alpha', \beta', v') \otimes (u, \alpha, \beta, v)\big) \\
&\quad = \big(- \downarrow_C -\big)\big((u', \alpha') *_0 (u, \alpha), (\beta', v') *_0 (\beta, v)\big) \\
&\quad = \big(- \downarrow_C -\big)\big(u' *_0 u, \alpha *_c (\alpha' *_r u), (\beta' *_r v) *_c \beta, v' *_0 v\big) \\
&\quad = \big((u' *_0 u) *_0 p_1, \\
&\qquad\quad \big[(\beta' *_r v) *_c \beta) *_r p_2\big] *_c \gamma *_c \big[(\alpha *_c (\alpha' *_r u) *_r p_1)\big], \\
&\qquad\quad (v' *_0 v) *_0 p_2\big)\,.
\end{aligned}$$

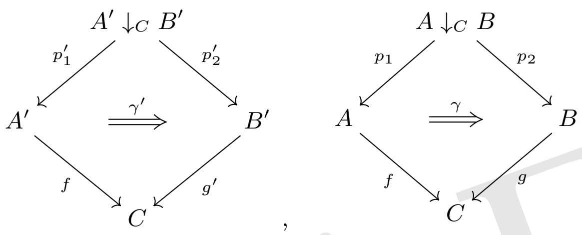
Note that this final formula is a categorification of the total composite of the diagram



Consider now the ω-functor N obtained as the composite

$$\begin{array}{c} \big(\underline{\mathsf{Hom}}_{/C}(A', A'') \times \underline{\mathsf{Hom}}^{\mathrm{to}}_{/C}(B', B'')\big) \otimes \big(\underline{\mathsf{Hom}}_{/C}(A, A') \times \underline{\mathsf{Hom}}^{\mathrm{to}}_{/C}(B, B')\big) \\ \Big\downarrow {\scriptstyle (-\downarrow_C -)\otimes(-\downarrow_C -)} \\ \underline{\mathsf{Hom}}_{\mathrm{ol}}(A' \downarrow_C B', A'' \downarrow_C B'') \otimes \underline{\mathsf{Hom}}_{\mathrm{ol}}(A \downarrow_C B, A' \downarrow_C B') \\ \Big\downarrow {\scriptstyle \circ} \\ \underline{\mathsf{Hom}}_{\mathrm{ol}}(A \downarrow_C B, A'' \downarrow_C B'') \quad . \end{array}$$

Let us compute $N\big((u', \alpha', \beta', v') \otimes (u, \alpha, \beta, v)\big)$. Denote by



the two universal 2-squares involved. By definition, we have

$$\begin{aligned} (- \downarrow_C -)(u', \alpha', \beta', v') &= (u' *_0 p'_1, (\beta' *_r p'_2) *_c \gamma' *_c (\alpha' *_r p'_1), v' *_0 p'_2)\,, \\ (- \downarrow_C -)(u, \alpha, \beta, v) &= (u *_0 p_1, (\beta *_r p_2) *_c \gamma *_c (\alpha *_r p_1), v *_0 p_2)\,, \end{aligned}$$

and we have to compute the composition of these two cells in $\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$. We have

$$\begin{aligned} &\Big(u' *_0 p'_1, (\beta' *_r p'_2) *_c \gamma' *_c (\alpha' *_r p'_1), v' *_0 p'_2\Big) \\ &\quad *_0 \Big(u *_0 p_1, (\beta *_r p_2) *_c \gamma *_c (\alpha *_r p_1), v *_0 p_2\Big) \\ &\qquad = \Big(u' *_0 (u *_0 p_1), \\ &\qquad\quad \big[\beta' *_r (v *_0 p_2)\big] *_c \big[(\beta *_r p_2) *_c \gamma *_c (\alpha *_r p_1)\big] *_c \big[\alpha' *_r (u *_0 p_1)\big], \\ &\qquad\quad v' *_0 (v *_0 p_2)\Big)\,. \end{aligned}$$

The axiom we are checking is thus equivalent to the equalities

$$u' *_0 (u *_0 p_1) = (u' *_0 u) *_0 p_1\,, \qquad v' *_0 (v *_0 p_2) = (v' *_0 v) *_0 p_2$$

and

$$\begin{aligned} &\big[\beta' *_r (v *_0 p_2)\big] *_c \big[(\beta *_r p_2) *_c \gamma *_c (\alpha *_r p_1)\big] *_c \big[\alpha' *_r (u *_0 p_1)\big] \\ &\quad = \big[(\beta' *_r v) *_c \beta) *_r p_2\big] *_c \gamma *_c \big[(\alpha *_c (\alpha' *_r u)) *_r p_1\big]\,. \end{aligned}$$

The two first equalities are obviously true. As for the last one, we have

$$\begin{aligned}
&\big[\beta' *_r (v *_0 p_2)\big] *_c \big[(\beta *_r p_2) *_c \gamma *_c (\alpha *_r p_1)\big] *_c \big[\alpha' *_r (u *_0 p_1)\big] \\
&\quad = \Big(\big[\beta' *_r (v *_0 p_2)\big] *_c \big[\beta *_r p_2\big]\Big) *_c \gamma *_c \Big(\big[\alpha *_r p_1\big] *_c \big[\alpha' *_r (u *_0 p_1)\big]\Big) \\
&\quad = \Big(\big[(\beta' *_r v) *_r p_2)\big] *_c \big[\beta *_r p_2\big]\Big) *_c \gamma *_c \Big(\big[\alpha *_r p_1\big] *_c \big[(\alpha' *_r u) *_r p_1\big]\Big) \\
&\quad = \big[(\beta' *_r v) *_c \beta) *_r p_2\big] *_c \gamma *_c \big[(\alpha *_c (\alpha' *_r u)) *_r p_1\big]\,,
\end{aligned}$$

where the first equality follows from the associativity of $*_c$, the second from the fact that $*_r$ is a right module action (see 5.3) and the last one from the fact that this action is compatible with $*_c$ (see the last square of 5.6). □

Remark 6.4. — If A and B are two fixed ω-categories, there is a canonical embedding

$$\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,C)^{\circ} \times \underline{\mathsf{Hom}}_{\mathrm{oplax}}(B,C)^{\mathrm{to}} \hookrightarrow \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C \times \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \overset{\mathrm{to}}{/} C\,.$$

Indeed, by Proposition 5.13, the ω-category $\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,C)^{\circ}$ canonically embeds in $\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C$, and, by duality, this implies that

$$\big(\underline{\mathsf{Hom}}_{(\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}})^{\mathrm{to}}}(A,C)^{\circ}\big)^{\mathrm{to}} = \underline{\mathsf{Hom}}_{(\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}})^{\mathrm{to}}}(A,C)^{\mathrm{t}} = \underline{\mathsf{Hom}}_{\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}}(A,C)^{\mathrm{to}}$$

embeds in $\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \overset{\mathrm{to}}{/} C$.

The comma construction can thus be restricted to a Gray ω-functor

$$- \downarrow_C - \colon \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,C)^{\circ} \times \underline{\mathsf{Hom}}_{\mathrm{oplax}}(B,C)^{\mathrm{to}} \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\,.$$

Remark 6.5. — By duality, one can deduce that the *lax* comma construction $- \downarrow' -$ defines a *skew* Gray ω-functor. This follows from the formula $A \downarrow'_C B \simeq \big(A^{\mathrm{co}} \downarrow_{C^{\mathrm{co}}} B^{\mathrm{co}}\big)^{\mathrm{co}}$ of 4.7. Indeed, by 3.23, the duality D_{co} defines an isomorphism of skew Gray ω-categories $D_{\mathrm{co}} \colon (\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}})^{\mathrm{top}} \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}$ and we can consider the chain of skew Gray ω-functors and skew Gray ω-functors

$$\begin{array}{l}
\big((\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}})^{\mathrm{top}}/C\big)^{\mathrm{top}} \times \big((\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}})^{\mathrm{top}} \overset{\mathrm{to}}{/} C\big)^{\mathrm{top}} \\
\qquad\qquad\qquad\qquad {\scriptstyle D_{\mathrm{co}}}\big\downarrow \\
\big(\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C^{\mathrm{co}}\big)^{\mathrm{top}} \times \big(\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \overset{\mathrm{to}}{/} C^{\mathrm{co}}\big)^{\mathrm{top}} \xrightarrow{-\downarrow_{C^{\mathrm{co}}}-} (\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}})^{\mathrm{top}} \xrightarrow{D_{\mathrm{co}}} \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}\,.
\end{array}$$

Composing this chain, we get a skew Gray ω-functor

$$- \downarrow'_C - \colon \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}} \overset{\mathrm{top}}{/} C \times \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}} \overset{\mathrm{co}}{/} C \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}\,.$$

Corollary 6.6. — *If $B \to C$ is an ω-functor, then we have a Gray ω-functor*

$$- \downarrow_C B \colon \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\,,$$

and if $A \to C$ is an ω-functor, then we have a Gray ω-functor

$$A \downarrow_C - \colon \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \overset{\mathrm{to}}{/} C \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\,.$$

Remark 6.7. — The two Gray ω-functors of the previous corollary can be deduced from one another using the formula $A \downarrow_C B \simeq \big(B^\circ \downarrow_{C^\circ} A^\circ\big)^\circ$ of 4.7. For instance, if $u \colon A \to C$ is an ω-functor, the Gray ω-functor $A \downarrow_C -$ can be identified to the composition of the Gray ω-functors

$$\big((\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}})^{\mathrm{to}}/C\big)^{\mathrm{to}} \xrightarrow{D_\circ} \big(\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C^\circ\big)^{\mathrm{to}} \xrightarrow{-\downarrow_{C^\circ} A^\circ} (\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}})^{\mathrm{to}} \xrightarrow{D_\circ} \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \quad .$$

We end the section by expressing that the comma construction of $A \downarrow_C B$ is above A and B with a natural transformation.

Proposition 6.8. — *Let C be an ω-category. The canonical projection*

$$p = (p_1, p_2) \colon A \downarrow_C B \to A \times B$$

is natural in

$$A \longrightarrow C \longleftarrow B\,,$$

in the sense that it defines a natural transformation (in an enriched sense) from the Gray ω-functor

$$- \downarrow_C - \colon \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C \times \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \overset{\mathrm{to}}{/} C \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$$

to the Gray ω-functor obtained by composing

$$\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C \times \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \overset{\mathrm{to}}{/} C \xrightarrow{U \times U} \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \times \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \xrightarrow{\times} \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \quad ,$$

where U denotes the forgetful Gray ω-functor (see 5.11) and $\times$ is the product Gray ω-functor (see 2.6).

Proof. — Let A, A', B and B' four ω-categories above C. Using the same abbreviation as in the proof of Theorem 6.3, we have to show the commutativity of the square

$$\begin{array}{ccc}
\underline{\mathsf{Hom}}_{/C}(A, A') \times \underline{\mathsf{Hom}}_{\overset{\mathrm{to}}{/C}}(B, B') & \xrightarrow{-\downarrow_C -} & \underline{\mathsf{Hom}}_{\mathrm{ol}}(A \downarrow_C B, A' \downarrow_C B') \\
{\scriptstyle \times}\big\downarrow & & \big\downarrow{\scriptstyle \underline{\mathsf{Hom}}_{\mathrm{ol}}(A \downarrow_C B, p')} \\
\underline{\mathsf{Hom}}_{\mathrm{ol}}(A \times B, A' \times B') & \xrightarrow[\underline{\mathsf{Hom}}_{\mathrm{ol}}(p, A' \downarrow_C B')]{} & \underline{\mathsf{Hom}}_{\mathrm{ol}}(A \downarrow_C B, A' \times B')
\end{array} \quad ,$$

where we denoted simply by $\times$ the target Gray ω-functor of the statement. So let (u, α, β, v) be a cell in the source ω-category of this square. Using the formula defining the Gray comma construction, we get that this cell is sent to $(u *_0 p_1, v *_0 p_2)$ in $\underline{\mathsf{Hom}}_{\mathrm{ol}}(A \downarrow_C B, A' \times B')$ by the upper path of the square. But the Gray ω-functor $\times$ send this same cell to $(u *_0 q_1, v *_0 q_2)$ in $\underline{\mathsf{Hom}}_{\mathrm{ol}}(A \times B, A' \times B')$, where $q_1 \colon A \times B \to A$ and $q_2 \colon A \times B \to B$ are the two projections, and since

$$(u *_0 q_1, v *_0 q_2) *_0 p = (u *_0 q_1 *_0 p, v *_0 q_2 *_0 p) = (u *_0 p_1, v *_0 p_2)\,,$$

the square indeed commutes. □

7. Strict Functorialities of the comma construction

The purpose of this section is to study the functorialities of the comma construction when restricted to strict k-transformations.

7.1. — Fix C an ω-category. The inclusion $\omega\text{-}\mathcal{C}at_{\mathrm{cart}} \hookrightarrow \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$ induces inclusions

$$\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C \hookrightarrow \omega\text{-}\mathcal{C}at_{\mathrm{cart}}/C \quad \text{and} \quad \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\overset{\mathrm{to}}{/}C \hookrightarrow \omega\text{-}\mathcal{C}at_{\mathrm{cart}}\overset{\mathrm{to}}{/}C\,.$$

In particular, we can restrict the Gray ω-functor

$$-\downarrow_C - \colon \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C \times \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\overset{\mathrm{to}}{/}C \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$$

to a Gray ω-functor

$$-\downarrow_C - \colon \omega\text{-}\mathcal{C}at_{\mathrm{cart}}/C \times \omega\text{-}\mathcal{C}at_{\mathrm{cart}}\overset{\mathrm{to}}{/}C \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\,.$$

The purpose of this section is to prove that this Gray ω-functor actually lands in $\omega\text{-}\mathcal{C}at_{\mathrm{cart}}$.

The strategy is obvious. We gave in 6.2 a formula for this Gray ω-functor and it suffices to check that the formula defines a cell of $\omega\text{-}\mathcal{C}at_{\mathrm{cart}}$. But this formula involves the oplax transformation γ of the universal 2-square which does not live in $\omega\text{-}\mathcal{C}at_{\mathrm{cart}}$! Nevertheless, we will see that the result is indeed in $\omega\text{-}\mathcal{C}at_{\mathrm{cart}}$. To do so, we will introduce an intermediate slice ω-category

$$\omega\text{-}\mathcal{C}at_{\mathrm{cart}}/C \hookrightarrow \omega\text{-}\mathcal{C}at_{\mathrm{cart}}/_{\Gamma}C \hookrightarrow \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C$$

based on the inclusions

$$\Gamma\underline{\mathsf{Hom}}(A,B) \hookrightarrow \underline{\mathsf{Hom}}(A,\Gamma B) \hookrightarrow \Gamma\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,B)$$

of 3.18.

7.2. — Let A and B be two ω-categories. As mentioned above, we defined in 3.18 inclusions

$$\Gamma\underline{\mathsf{Hom}}(A,B) \hookrightarrow \underline{\mathsf{Hom}}(A,\Gamma B) \hookrightarrow \Gamma\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,B)$$

factorising the canonical inclusion.

If B is fixed and A varies, we get inclusions

$$\Gamma\underline{\mathsf{Hom}}(-,B) \hookrightarrow \underline{\mathsf{Hom}}(-,\Gamma B) \hookrightarrow \Gamma\underline{\mathsf{Hom}}_{\mathrm{oplax}}(-,B)$$

of skew Gray ω-functors from $(\omega\text{-}\mathcal{C}at_{\mathrm{cart}})^{\mathrm{t}}$ to $\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}$, and hence by Proposition 2.10, morphisms of right $\omega\text{-}\mathcal{C}at_{\mathrm{cart}}$-modules. In particular, this means that if α is a cell of $\underline{\mathsf{Hom}}(A',\Gamma B)$ and u is a cell of $\underline{\mathsf{Hom}}(A,A')$, then $\alpha *_r u$, where $*_r$ denotes the right action of 5.3, is a cell of $\underline{\mathsf{Hom}}(A,\Gamma B)$.

Moreover, for the same reasons as in 5.5, each of the three ω-categories

$$\Gamma\underline{\mathsf{Hom}}(A,B) \hookrightarrow \underline{\mathsf{Hom}}(A,\Gamma B) \hookrightarrow \Gamma\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,B)$$

is the object of morphisms of a category internal to $\omega\text{-}\mathcal{C}at$ and, by naturality, the two inclusions are morphisms of internal categories. In particular, if α and β are i-cells of $\underline{\mathsf{Hom}}(A,\Gamma B)$ such that $\mathbb{t}(\beta) = \mathbb{s}(\alpha)$, then $\beta *_c \alpha$, where $*_c$ denotes the composition of cylinders of 3.7, is an i-cell of $\underline{\mathsf{Hom}}(A,\Gamma B)$.

7.3. — Fix C an ω-category. We define an ω-category $\omega\text{-}\mathcal{C}at_{\mathrm{cart}}/_{\Gamma}C$ in the following way.

The objects of $\omega\text{-}\mathcal{C}at_{\mathrm{cart}}/_{\Gamma}C$ are the same as the ones of $\omega\text{-}\mathcal{C}at_{\mathrm{cart}}/C$ or of $\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C$, that is, the ω-categories A endowed with an ω-functor $f\colon A \to C$.

If $(A, f\colon A \to C)$ and $(A', f'\colon A' \to C)$ are two such objects, we set

$$\begin{aligned}&\underline{\mathsf{Hom}}_{\omega\text{-}\mathcal{C}at/_{\Gamma}C}((A,f),(A',f'))\\ &\qquad = \underline{\mathsf{Hom}}(A,A') \times_{\underline{\mathsf{Hom}}(A,C)} \underline{\mathsf{Hom}}(A,\Gamma C) \times_{\underline{\mathsf{Hom}}(A,C)} \{f\}\,.\end{aligned}$$

For the moment, $\omega\text{-}\mathcal{C}at_{\mathrm{cart}}/_{\Gamma}C$ has only been defined as a graph enriched in $\omega\text{-}\mathcal{C}at$.

Recall that

$$\begin{aligned}&\underline{\mathsf{Hom}}_{\omega\text{-}\mathcal{C}at_{\mathrm{cart}/C}}((A,f),(A',f'))\\ &\qquad = \underline{\mathsf{Hom}}(A,A') \times_{\underline{\mathsf{Hom}}(A,C)} \Gamma\underline{\mathsf{Hom}}(A,C) \times_{\underline{\mathsf{Hom}}(A,C)} \{f\}\end{aligned}$$

and

$$\begin{aligned}&\underline{\mathsf{Hom}}_{\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}/C}}((A,f),(A',f'))\\ &\qquad = \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,A') \times_{\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,C)} \Gamma\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,C) \times_{\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,C)} \{f\}\,.\end{aligned}$$

Therefore, the monomorphisms

$$\Gamma\underline{\mathsf{Hom}}(A,C) \hookrightarrow \underline{\mathsf{Hom}}(A,\Gamma C) \hookrightarrow \Gamma\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,C)\,,$$

as they are compatible with the source and target operations of internal categories, induce monomorphisms between these fiber products and hence monomorphisms of graphs enriched in $\omega\text{-}\mathcal{C}at$

$$\omega\text{-}\mathcal{C}at_{\mathrm{cart}}/C \hookrightarrow \omega\text{-}\mathcal{C}at_{\mathrm{cart}}/_{\Gamma}C \hookrightarrow \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C \quad .$$

We will consider these monomorphisms as inclusions. Note that $\omega\text{-}\mathcal{C}at_{\mathrm{cart}}/_{\Gamma}C$ have not only the same objects as $\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C$ but also the same 1-cells. (But their i-cells differ for $i > 1$.)

Proposition 7.4. — *If C is an ω-category, then $\omega\text{-}\mathcal{C}at_{\mathrm{cart}}/_{\Gamma}C$ is a sub-Gray ω-category of $\omega\text{-}\mathcal{C}at/C$. It it actually a strict ω-category.*

Proof. — Let $(A, f\colon A \to C)$, $(A', f'\colon A' \to C)$ and $(A'', f''\colon A'' \to C)$ be three objects of $\omega\text{-}\mathcal{C}at/_{\Gamma}C$ and let (u,α) be a cell of $\underline{\mathsf{Hom}}_{\omega\text{-}\mathcal{C}at/_{\Gamma}C}((A,f),(A',f'))$ and (u',α') a cell of $\underline{\mathsf{Hom}}_{\omega\text{-}\mathcal{C}at/_{\Gamma}C}((A',f'),(A'',f''))$. By definition, their composition in $\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C$ is given by

$$(u',\alpha') *_0 (u,\alpha) = \big(u' *_0 u, \alpha *_c (\alpha' *_r u)\big)\,,$$

where $*_r$ denotes the right action of 5.3 and $*_c$ denotes the internal composition on $\Gamma\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A,C)$ of 3.7. Since $\omega\text{-}\mathcal{C}at_{\mathrm{cart}}$ is a sub-Gray ω-category of $\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$, the cell $u' *_0 u$ lives in $\omega\text{-}\mathcal{C}at_{\mathrm{cart}}$. Moreover, by 7.3, the cell $\alpha' *_r u$ is in $\underline{\mathsf{Hom}}(A,\Gamma C)$ and hence so is $\alpha *_c (\alpha' *_r u)$, thereby proving the stability by composition of $\omega\text{-}\mathcal{C}at_{\mathrm{cart}}/_{\Gamma}C$. The compatibility with units is obvious.

The fact that $\omega\text{-}\mathcal{C}at_{\mathrm{cart}}/_{\Gamma}C$ is a strict ω-category follows from the formula giving the composition and the fact that $\omega\text{-}\mathcal{C}at_{\mathrm{cart}}$ is a strict ω-category. □

7.5. — Let C be an ω-category. One defines similarly an ω-category $\omega\text{-}\mathcal{C}at_{\mathrm{cart}}{\overset{\mathrm{to}}{/}}_{\Gamma} C$ with Gray inclusions

$$\omega\text{-}\mathcal{C}at_{\mathrm{cart}}\overset{\mathrm{to}}{/}C \hookrightarrow \omega\text{-}\mathcal{C}at_{\mathrm{cart}}{\overset{\mathrm{to}}{/}}_{\Gamma}C \hookrightarrow \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\overset{\mathrm{to}}{/}C \quad .$$

The objects of $\omega\text{-}\mathcal{C}at_{\mathrm{cart}}{\overset{\mathrm{to}}{/}}_{\Gamma}C$ are the ω-categories B endowed with an ω-functor $g\colon B \to C$, and if $(B, g\colon B \to C)$ and $(B', g'\colon B' \to C)$ are two such objects, we have

$$\begin{aligned}&\underline{\mathsf{Hom}}_{\omega\text{-}\mathcal{C}at{\overset{\mathrm{to}}{/}}_{\Gamma}C}((B,g),(B',g')) \\ &\qquad = \{g\} \times_{\underline{\mathsf{Hom}}(B,C)} \underline{\mathsf{Hom}}(B,\Gamma C) \times_{\underline{\mathsf{Hom}}(B,C)} \underline{\mathsf{Hom}}(B,B')\,.\end{aligned}$$

Proposition 7.6. — *Let C be an ω-category. The Gray ω-functor*

$$- \downarrow_C - \colon \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}/C \times \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\overset{\mathrm{to}}{/}C \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$$

induces an ω-functor

$$- \downarrow_C - \colon \omega\text{-}\mathcal{C}at_{\mathrm{cart}/_{\Gamma}}C \times \omega\text{-}\mathcal{C}at_{\mathrm{cart}}{\overset{\mathrm{to}}{/}}_{\Gamma}C \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}$$

and in particular an ω-functor

$$- \downarrow_C - \colon \omega\text{-}\mathcal{C}at_{\mathrm{cart}}/C \times \omega\text{-}\mathcal{C}at_{\mathrm{cart}}\overset{\mathrm{to}}{/}C \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}\,.$$

Proof. — Consider A, A', B, B' four ω-categories over C. We have to show that the composite ω-functor

$$\underline{\mathsf{Hom}}_{\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}/_{\Gamma}}C}((A,f),(A',f')) \times \underline{\mathsf{Hom}}_{\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}{\overset{\mathrm{to}}{/}}_{\Gamma}C}((B,g),(B',g'))$$

$$\Big\downarrow$$

$$\underline{\mathsf{Hom}}_{\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}/C}}((A,f),(A',f')) \times \underline{\mathsf{Hom}}_{\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\overset{\mathrm{to}}{/}C}((B,g),(B',g'))$$

$$\Big\downarrow{\scriptstyle -\downarrow_C-}$$

$$\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A \downarrow_C B, A') \underset{\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A\downarrow_C B, C)}{\downarrow} \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A \downarrow_C B, B')$$

$$\simeq$$

$$\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A \downarrow_C B, A' \downarrow_C B')$$

factors through

$$\underline{\mathsf{Hom}}(A \downarrow_C B, A' \downarrow_C B')\,.$$

Using 6.2, and with its notation, this ω-functor is given on i-cells by

$$(u, \alpha, \beta, v) \mapsto (u *_0 p_1, (\beta *_r p_2) *_c \gamma *_c (\alpha *_r p_1), v *_0 p_2)\,,$$

where

$$(u, \alpha, \beta, v) \quad \text{is in} \quad \underline{\mathsf{Hom}}(A, A')_i \times \underline{\mathsf{Hom}}(A, \Gamma C)_i \times \underline{\mathsf{Hom}}(B, \Gamma C)_i \times \underline{\mathsf{Hom}}(B, B')_i\,.$$

Note that γ can be seen as a 0-cell of $\underline{\mathsf{Hom}}(A \downarrow_C B, \Gamma C)$. Using 7.3, we get that

$$(\beta *_r p_2) *_c \gamma *_c (\alpha *_r p_1)$$

is actually an i-cell in $\underline{\mathsf{Hom}}(A \downarrow_C B, \Gamma C)$, and in particular,

$$(u *_0 p_1, (\beta *_r p_2) *_c \gamma *_c (\alpha *_r p_1), v *_0 p_2)$$

is an i-cell of

$$\underline{\mathsf{Hom}}(A \downarrow_C B, A') \underset{\underline{\mathsf{Hom}}(A \downarrow_C B, C)}{\times} \underline{\mathsf{Hom}}(A \downarrow_C B, \Gamma C) \underset{\underline{\mathsf{Hom}}(A \downarrow_C B, C)}{\times} \underline{\mathsf{Hom}}(A \downarrow_C B, B')$$

which is canonically isomorphic to

$$\underline{\mathsf{Hom}}(A \downarrow_C B, A' \times_C \Gamma C \times_C B') \simeq \underline{\mathsf{Hom}}(A \downarrow_C B, A' \downarrow_C B') \,.$$

As this canonical isomorphism is compatible with the canonical isomorphism between

$$\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A \downarrow_C B, A') \underset{\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A \downarrow_C B, C)}{\times} \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A \downarrow_C B, \Gamma C) \underset{\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A \downarrow_C B, C)}{\times} \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A \downarrow_C B, B')$$

and

$$\underline{\mathsf{Hom}}_{\mathrm{oplax}}(A \downarrow_C B, A' \times_C \Gamma C \times_C B') \simeq \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A \downarrow_C B, A' \downarrow_C B') \,,$$

this proves the result. □

Remark 7.7. — As in Remark 6.4, if A and B are two fixed ω-categories, there is a canonical embedding

$$\underline{\mathsf{Hom}}(A, C)^{\circ} \times \underline{\mathsf{Hom}}(B, C)^{\mathrm{to}} \hookrightarrow \omega\text{-}\mathcal{C}at_{\mathrm{cart}/C} \times \omega\text{-}\mathcal{C}at_{\mathrm{cart}} \overset{\mathrm{to}}{/} C$$

and the comma construction thus restricts to an ω-functor

$$- \downarrow_C - \colon \underline{\mathsf{Hom}}(A, C)^{\circ} \times \underline{\mathsf{Hom}}(B, C)^{\mathrm{to}} \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}} \,.$$

Similarly, if A is ω-category, let us denote by $\underline{\mathsf{Hom}}_{\Gamma}(A, B)$ the total dual of the fiber at A of the forgetful ω-functor

$$U \colon \omega\text{-}\mathcal{C}at_{\mathrm{cart}/_{\Gamma}C} \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}} \,.$$

Note that we have inclusions

$$\underline{\mathsf{Hom}}(A, C) \hookrightarrow \underline{\mathsf{Hom}}_{\Gamma}(A, C) \hookrightarrow \underline{\mathsf{Hom}}_{\mathrm{oplax}}(A, C) \,.$$

By definition (and duality), we have a canonical embedding

$$\underline{\mathsf{Hom}}_{\Gamma}(A, C)^{\circ} \times \underline{\mathsf{Hom}}_{\Gamma}(B, C)^{\mathrm{to}} \hookrightarrow \omega\text{-}\mathcal{C}at_{\mathrm{cart}/C} \times \omega\text{-}\mathcal{C}at_{\mathrm{cart}} \overset{\mathrm{to}}{/} C$$

and the comma construction also restricts to an ω-functor

$$- \downarrow_C - \colon \underline{\mathsf{Hom}}_{\Gamma}(A, C)^{\circ} \times \underline{\mathsf{Hom}}_{\Gamma}(B, C)^{\mathrm{to}} \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}} \,.$$

Remark 7.8. — By duality (see Remark 6.5), Proposition 7.6 implies that the lax comma construction induce an ω-functor

$$- \downarrow'_C - \colon \omega\text{-}\mathcal{C}at_{\mathrm{cart}} \overset{\mathrm{top}}{/} C \times \omega\text{-}\mathcal{C}at_{\mathrm{cart}} \overset{\mathrm{co}}{/} C \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}\,,$$

and, more generally, an ω-functor

$$- \downarrow'_C - \colon \omega\text{-}\mathcal{C}at_{\mathrm{cart}} \overset{\mathrm{top}}{/_{\Gamma'}} C \times \omega\text{-}\mathcal{C}at_{\mathrm{cart}} \overset{\mathrm{co}}{/_{\Gamma'}} C \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}\,,$$

where the ω-category

$$\omega\text{-}\mathcal{C}at_{\mathrm{cart}} \overset{\mathrm{top}}{/_{\Gamma'}} C \quad \text{and} \quad \omega\text{-}\mathcal{C}at_{\mathrm{cart}} \overset{\mathrm{co}}{/_{\Gamma'}} C$$

are defined from their undecorated analogue by replacing in the definition of the ω-category of morphisms the construction $\Gamma' \,\underline{\mathsf{Hom}}(-, C)$ by $\underline{\mathsf{Hom}}(-, \Gamma' C)$. (Recall that $\Gamma' X = \underline{\mathsf{Hom}}_{\mathrm{oplax}}(\mathrm{D}_1, X)$.)

Corollary 7.9. — *If $B \to C$ is an ω-functor, then we have an ω-functor*

$$- \downarrow_C B \colon \omega\text{-}\mathcal{C}at_{\mathrm{cart}}/_{\Gamma} C \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}$$

and in particular an ω-functor

$$- \downarrow_C B \colon \omega\text{-}\mathcal{C}at_{\mathrm{cart}}/C \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}\,,$$

and if $A \to C$ is an ω-functor, then we have an ω-functor

$$A \downarrow_C - \colon \omega\text{-}\mathcal{C}at_{\mathrm{cart}} \overset{\mathrm{to}}{/_{\Gamma}} C \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}$$

and in particular an ω-functor

$$A \downarrow_C - \colon \omega\text{-}\mathcal{C}at_{\mathrm{cart}} \overset{\mathrm{to}}{/} C \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}\,.$$

8. Application: Grothendieck construction for ω-categories

Our main motivation for studying the functorialities of the comma construction was the Grothendieck construction for ω-categories, to which we will devote a whole paper [**2**]. In this short final section, we define the Grothendieck construction for ω-categories in terms of comma ω-categories and we deduce functoriality results for the Grothendieck construction.

8.1. — Let I be an ω-category and let $F \colon I^\circ \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}$ be an ω-functor. We define the *(contravariant) Grothendieck construction* $\int_I^\circ F$ of F to be the total dual of the comma construction of the diagram

$$\mathrm{D}_0 \xrightarrow{\mathrm{D}_0} \omega\text{-}\mathcal{C}at_{\mathrm{cart}} \xleftarrow{F} I^\circ \quad ,$$

where the left arrow corresponds to the object D_0 of $\omega\text{-}\mathcal{C}at_{\mathrm{cart}}$. In other words, we have

$$\textstyle\int_I^\circ F = \left(\mathrm{D}_0 \downarrow_{\omega\text{-}\mathcal{C}at_{\mathrm{cart}}} F\right)^\circ.$$

The second projection of the comma construction induces an ω-functor $p \colon \int_I^\circ F \to I$.

Remark 8.2. — Notes that although the ω-category $\omega\text{-}\mathcal{C}at_{\mathrm{cart}}$ is not small, the comma construction $\mathrm{D}_0 \downarrow F$ makes sense and is a small ω-category.

Remark 8.3. — By Example 4.2, the Grothendieck construction of an ω-functor $F\colon I^\circ \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}$ is the total dual of a relative slice:

$$\textstyle\int_I^\circ F = \big(\mathrm{D}_0 \downarrow F\big)^\circ = \big(\mathrm{D}_0\backslash\omega\text{-}\mathcal{C}at_{\mathrm{cart}}\big)^\circ .$$

In other words, we have a pullback square

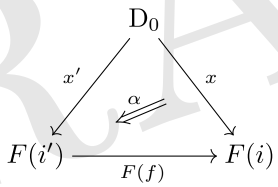
$$\begin{array}{ccc} (\int_I^\circ F)^\circ & \longrightarrow & \mathrm{D}_0\backslash\omega\text{-}\mathcal{C}at_{\mathrm{cart}} \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle U} \\ I^\circ & \xrightarrow[F]{} & \omega\text{-}\mathcal{C}at_{\mathrm{cart}} \end{array} ,$$

where U denotes the forgetful ω-functor.

8.4. — Let $F\colon I^\circ \to \omega\text{-}\mathcal{C}at$ be an ω-functor. To convince ourself that our definition of the Grothendieck construction is reasonable, let us describe concretely the cells of $\int_I^\circ F$ in low dimensions.

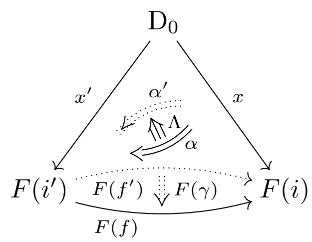
By definition, an object of $\int_I^\circ F$ corresponds to an object i of I and an ω-functor $x\colon \mathrm{D}_0 \to F(i)$. These objects can thus be identified with pairs (i, x), where i is an object of I and x an object of $F(i)$.

A 1-cell of $\int_I^\circ F$ corresponds to a 1-cell $f\colon i \to i'$ of I and a 2-triangle



in $\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$. But the data of such an oplax transformation α is equivalent to the data of a 1-cell $\alpha\colon x \to F(f)(x')$ of $F(i)$. The 1-cells from (i, x) to (i', x') can thus be identified with pairs $(f\colon i \to i', \alpha\colon x \to F(f)(x'))$.

A 2-cell of $\int_I^\circ F$ corresponds to 2-cell $\gamma\colon f \Rightarrow f'\colon i \to i'$ of I and a 3-cone



in $\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$. But the data of such an oplax 2-transformation Λ is equivalent to the data of a 2-cell $\Lambda\colon \alpha \Rightarrow F(\gamma)_{x'} *_0 \alpha'$ of $F(i)$. The 2-cells from (f, α) to (f', α') can thus be identified with these pairs (γ, Λ).

Remark 8.5. — The Grothendieck construction for ω-categories was first defined by Warren [**8**] using explicit formulas. In [**2**], we will show that our definition is equivalent to Warren's one (up to some duality, as Warren defines the *covariant* Grothendieck construction).

8.6. — We will denote by $\omega\text{-}\mathbb{C}\mathrm{AT}_{\mathrm{oplax}}$ the (very large) Gray ω-category of possibly large ω-categories, ω-functors, oplax transformations and higher oplax transformations between them. We have a fully faithful inclusion $\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \hookrightarrow \omega\text{-}\mathbb{C}\mathrm{AT}_{\mathrm{oplax}}$. The ω-category $\omega\text{-}\mathcal{C}at_{\mathrm{cart}}$ is an object of $\omega\text{-}\mathbb{C}\mathrm{AT}_{\mathrm{oplax}}$. When we consider $\omega\text{-}\mathcal{C}at_{\mathrm{cart}}$ as an object of $\omega\text{-}\mathbb{C}\mathrm{AT}_{\mathrm{oplax}}$, we denote it by $\{\omega\text{-}\mathcal{C}at_{\mathrm{cart}}\}$. In particular, we will write

$$\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \overset{\mathrm{to}}{/} \{\omega\text{-}\mathcal{C}at_{\mathrm{cart}}\}$$

for the Gray ω-category defined by the pullback

$$\begin{array}{ccc} \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \overset{\mathrm{to}}{/} \{\omega\text{-}\mathcal{C}at_{\mathrm{cart}}\} & \longrightarrow & \omega\text{-}\mathbb{C}\mathrm{AT}_{\mathrm{oplax}} \overset{\mathrm{to}}{/} \{\omega\text{-}\mathcal{C}at_{\mathrm{cart}}\} \\ \downarrow & \lrcorner & \downarrow{\scriptstyle U} \\ \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} & \hookrightarrow & \omega\text{-}\mathbb{C}\mathrm{AT}_{\mathrm{oplax}} \end{array} ,$$

where $\omega\text{-}\mathbb{C}\mathrm{AT}_{\mathrm{oplax}} \overset{\mathrm{to}}{/} \{\omega\text{-}\mathcal{C}at_{\mathrm{cart}}\}$ is one of the (large) slice Gray ω-categories of 5.14 and U is the forgetful Gray ω-functor.

Theorem 8.7. — *The Grothendieck construction defines a Gray ω-functor*

$$\begin{aligned} \textstyle\int^\circ \colon \big(\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \overset{\mathrm{to}}{/} \{\omega\text{-}\mathcal{C}at_{\mathrm{cart}}\}\big)^{\mathrm{to}} &\to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \\ F \colon I^\circ \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}} &\mapsto \textstyle\int_I^\circ F \end{aligned} .$$

Proof. — The Grothendieck construction factors as

$$\big(\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \overset{\mathrm{to}}{/} \{\omega\text{-}\mathcal{C}at_{\mathrm{cart}}\}\big)^{\mathrm{to}} \xrightarrow{D_0 \downarrow -} \big(\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\big)^{\mathrm{to}} \xrightarrow{D_\circ} \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} .$$

But the left arrow is a Gray ω-functor by Theorem 6.3 (and more precisely Corollary 7.9) and the right arrow is a Gray ω-functor by 3.24. ☐

Proposition 8.8. — *The ω-functor $p\colon \int_I^\circ F \to I$ is natural in $F\colon I^\circ \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}$ in the sense that it defines a natural transformation (in the enriched sense) from the Gray ω-functor*

$$\textstyle\int^\circ \colon \big(\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \overset{\mathrm{to}}{/} \{\omega\text{-}\mathcal{C}at_{\mathrm{cart}}\}\big)^{\mathrm{to}} \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}$$

to the Gray ω-functor obtained by composing

$$\big(\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \overset{\mathrm{to}}{/} \{\omega\text{-}\mathcal{C}at_{\mathrm{cart}}\}\big)^{\mathrm{to}} \xrightarrow{U} \big(\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\big)^{\mathrm{to}} \xrightarrow{D_\circ} \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} ,$$

where D_o is the total dual.

Proof. — This is a particular case of the analogous result for the comma construction, that is, Proposition 6.8. ☐

Proposition 8.9. — *If I is a fixed ω-category, the Grothendieck construction restricts to a Gray ω-functor*

$$\textstyle\int_I^{\circ} \colon\ \underline{\mathsf{Hom}}_{\mathrm{oplax}}(I, \omega\text{-}\mathcal{C}at_{\mathrm{cart}}) \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}}\,.$$

Proof. — By Remark 6.4, there is a canonical embedding

$$\underline{\mathsf{Hom}}_{\mathrm{oplax}}(I, \omega\text{-}\mathcal{C}at_{\mathrm{cart}})^{\mathrm{to}} \hookrightarrow \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{oplax}} \overset{\mathrm{to}}{/} \{\omega\text{-}\mathcal{C}at_{\mathrm{cart}}\}\,,$$

hence the result, by the previous proposition. □

8.10. — We are now going to state the strict functorialities of the Grothendieck construction. We can define as in 8.6 a large strict ω-category

$$\begin{array}{ccc}
\omega\text{-}\mathcal{C}at_{\mathrm{cart}} \overset{\mathrm{to}}{/} \{\omega\text{-}\mathcal{C}at_{\mathrm{cart}}\} & \longrightarrow & \omega\text{-}\mathcal{C}\mathrm{AT}_{\mathrm{cart}} \overset{\mathrm{to}}{/} \{\omega\text{-}\mathcal{C}at_{\mathrm{cart}}\} \\
\big\downarrow & \lrcorner & \big\downarrow U \\
\omega\text{-}\mathcal{C}at_{\mathrm{cart}} & \hookrightarrow & \omega\text{-}\mathcal{C}\mathrm{AT}_{\mathrm{cart}}
\end{array}\quad,$$

where $\omega\text{-}\mathcal{C}\mathrm{AT}_{\mathrm{cart}}$ denotes the very large strict ω-category of possibly large ω-categories, ω-functors, strict transformations and higher strict transformations between them.

Proposition 8.11. — *The Grothendieck construction restricts to a strict ω-functor*

$$\textstyle\int^{\circ} \colon\ \big(\omega\text{-}\mathcal{C}at_{\mathrm{cart}} \overset{\mathrm{to}}{/} \{\omega\text{-}\mathcal{C}at_{\mathrm{cart}}\}\big)^{\mathrm{to}} \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}\,,$$

and, if I is a fixed ω-category, to a strict ω-functor

$$\textstyle\int_I^{\circ} \colon\ \underline{\mathsf{Hom}}(I, \omega\text{-}\mathcal{C}at_{\mathrm{cart}}) \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}\,.$$

Proof. — This follows from the analogous result from the comma construction, that is, Proposition 7.6. □

Remark 8.12. — By Remark 7.7 and using the same notation, the second ω-functor of the above proposition actually extends to an ω-functor

$$\textstyle\int_I^{\circ} \colon\ \underline{\mathsf{Hom}}_{\Gamma}(I, \omega\text{-}\mathcal{C}at_{\mathrm{cart}}) \to \omega\text{-}\mathcal{C}at_{\mathrm{cart}}\,.$$

We end the paper by an opening on a definition of the Grothendieck construction for Gray ω-categories, based on Remark 8.3.

8.13. — Let $\mathbb{C}$ be a Gray ω-category, so that $\mathbb{C}^{\circ}$ is skew Gray ω-category, and fix $F \colon \mathbb{C}^{\circ} \to \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}$ a skew Gray ω-functor. We define the Grothendieck construction $\int_{\mathbb{C}}^{\circ} F$ of F as the Gray ω-category obtained as the total dual of the pullback of skew Gray ω-category

$$\begin{array}{ccc}
(\int_{\mathbb{C}}^{\circ} F)^{\circ} & \longrightarrow & \mathrm{D}_0\backslash\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}} \\
\big\downarrow & \lrcorner & \big\downarrow U \\
\mathbb{C}^{\circ} & \xrightarrow[F]{} & \omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}
\end{array}\quad,$$

where $\mathrm{D}_0\backslash\omega\text{-}\mathbb{C}\mathsf{at}_{\mathrm{lax}}$ is one of the slice skew Gray ω-categories defined in 5.14 and U is the forgetful skew Gray ω-functor.

By Remark 8.3, in the case where $\mathbb{C}$ comes from a strict ω-category and the ω-functor F factors through $\omega\text{-}\mathcal{C}at_{\text{cart}} \hookrightarrow \omega\text{-}\mathbb{C}\mathsf{at}_{\text{lax}}$, we recover the Grothendieck construction defined in 8.1.

In future work, we plan to investigate this Grothendieck construction for Gray ω-categories.

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