ADVANCED MATHEMATICS 2023-2024
EXERCISES – WEEK 1

* Exercise 1

Which of the following subsets of $\mathbb{Q}$ is bounded below? Bounded above? Has a least upper bound? Has a greatest lower bound?

- $\{x \in \mathbb{Q} \,|\, 0 < x < 1\}$,
- $\{x \in \mathbb{Q} \,|\, 0 \leq x \leq 1\}$,
- $\{x \in \mathbb{Q} \,|\, 0 \leq x\}$,
- $\{x \in \mathbb{Q} \,|\, x \leq 0\}$,
- $\{1 - \frac{1}{n} \,|\, n \in \mathbb{N}, n \neq 0\}$.

Answer of exercise 1

The first two ones are both bounded below and bounded above, and both have a least upper bound (which is 0 in both cases) and a greatest lower bound (which is 1 in both cases).

The third one is bounded below and has a greatest lower bound (which is 0) but is not bounded above.

The fourth one is bounded above and has a least upper bound (which is 0) but is not bounded below.

The last one is bounded below and have a least upper bound (which is 0). It is also bounded above and has a least upper bound (which is 1).

* Exercise 2

Let $(S, \leq)$ be an ordered set. Consider the relation $\leq'$ on S defined by

$$x \leq' y \text{ if } y \leq x.$$

Is $(S, \leq')$ an ordered set?

Answer of exercise 2

$(S, \leq')$ is an ordered set.

- antisymmetry. Let $x, y \in S$ with $x \leq' y$ & $y \leq' x$. The first inequality and the definition of $\leq'$ gives us that $y \leq x$ and the second, $x \leq y$ respectively. By the antisymmetry of $\leq$ we get $x = y$. Therefore, $\leq'$ is also antisymmetric.
- transitivity. Now let $x, y, z \in S$ with $x \leq' y$ & $y \leq' z$. From this and by definition of $\leq'$, we get that $y \leq x \;\; z \leq y$. Therefore, $z \leq y \leq x$ and by transitivity of $\leq$ we get $z \leq x$. Again, by definition of $\leq'$ we get $x \leq' z$ thus $\leq'$ is transitive.
- totality. Let $x, y \in S$. Since $\leq$ is a total order, we have that either $x \leq y$ or $y \leq x$. This gives us that either $y \leq' x$ or $x \leq' y$, which means that $\leq'$ is a total order.

* Exercise 3

Adapt the definitions seen in the lecture to give the precise definitions of *lower bound*, *greatest lower bound*. State the *greatest lower bound property.*

Answer of exercise 3

Let $(S, \leq)$ be an ordered set and $A \subseteq S$ a subset of S. An element $x \in S$ is a lower bound of A if for every element a of A, we have

$$x \leq a.$$

If such a lower bound exists, we say that A is bounded below.

A lower bound x of A is a greatest lower bound (or infimum) if for any other lower bound x' of A, we have

$$x' \leq x.$$

We say that an ordered set $(S, \leq)$ has *the greatest lower bound property* if every non-empty, bounded below, subset A of S has a greatest lower bound.

** Exercise 4

Show that there is no rational number x such that $x^2 = 2$.

Answer of exercise 4

Suppose there exists a rational number x such that $x^2 = 2$. Then

$$x = \frac{m}{n},$$

for some $m, n \in \mathbb{N}$ with $\gcd(m, n) = 1$. We have that

$$x^2 = 2 \Rightarrow \frac{m^2}{n^2} = 2 \Rightarrow m^2 = 2n^2$$

which means that m^2 is an even number. By number theory we know that m should also be even, i.e. $m = 2k$ for some $k \in \mathbb{N}$. Replacing m in the last equality we get

$$(2k)^2 = 2n^2 \Rightarrow n^2 = 2k^2$$

which by the same reasoning means that n is an even number, i.e. $n = 2k'$ for some $k' \in \mathbb{N}$. Now it is clear that $\gcd(m, n) = 2$ which is a contradiction.

** Exercise 5

Consider the set $\mathbb{N}$ of natural integers with the usual order. Show that every bounded above, non-empty, subset $S \subseteq \mathbb{N}$ has a maximum element, that is an element $n \in S$ which is greater than all other elements of S.

What is the difference between this property and the least upper bound property seen in the lecture?

Answer of exercise 5

Yes it has: let S be a non-empty, bounded above, subset of $\mathbb{N}$ and let n be its number of elements. By hypothesis $n \neq 0$. Let us prove the property by induction on $n > 0$.

Base case: if $n = 1$, then the property is trivial.

Inductive case: Suppose that the desired property holds for all non-empty, bounded above subsets of $\mathbb{N}$ of size $n - 1$. Let $S \subseteq \mathbb{N}$ be a non-empty subset of size n. Since S is non-empty, take any $x \in S$. Consider now the set $S' = S \setminus \{x\}$. This is a set of size $n - 1$ and thus by the induction hypothesis, has a maximum, say x^*. Now, since $\mathbb{N}$ is a total order we have that either

$x \leq x^*$ or $x^* \leq x$. In the first case, x^* is the maximum of S and in the second case x is the maximum of S. In both cases, S has a maximum element.

By the induction principle any non-empty, bounded above subset S of $\mathbb{N}$ has a maximum. The difference with the least upper bound property is that the maximum element belongs to S, which is stronger than just being a least upper bound.

** Exercise 6

Show that an ordered set $(S, \leq)$ has the least upper bound property if and only if it has the greatest lower bound property. (*Hint:* For a subset $A \subseteq S$, consider the subset of all the lower bounds of A.)

Answer of exercise 6

We will show that if $(S, \leq)$ has the least upper bound property then it has the greatest lower bound property. The other direction is similar.

Let $A \subseteq S$ be a non-empty subset of S which is bounded below. Define the set

$$A^L = \{x \in S \mid x \text{ is a lower bound of } A\}.$$

Since we have assumed that A is bounded below, the set A^L is non-empty. Now, since A is non-empty, then by definition of A^L any $a \in A$ is an upper bound of A^L. Thus, A^L is a non-empty subset of S, bounded above. By the least upper bound property of S we get that A^L has a least upper bound, say x^*.

We show that x^* is the greatest lower bound of A. Indeed,

- it is a lower bound. Let $a \in A$. For any $x \in A^L$, $x \leq a$, thus a is an upper bound of A^L. But x^* is the least upper bound of A^L. Thus, $x^* \leq a$. Since $a \in A$ was arbitrary, we conclude that $x^* \leq a$ for every $a \in A$.
- it is the greatest lower bound of A. Indeed, let $x \in S$ be a lower bound of A. This is an element of A^L by definition of the latter and since x^* is an upper bound of A^L we get that $x \leq x^*$.

** Exercise 7

Let $x > 0$ be a real number. Show that there exists a largest integer n such that $n \leq x$. This integer is usually denoted as $[x]$ and referred to as the *integral part of* x. (*Hint:* Use the archimedean property of $\mathbb{R}$ and Exercise 5.)

Answer of exercise 7

Consider the set

$$A = \{n \in \mathbb{N} \mid n \leq x\}.$$

By the archimedean property of $\mathbb{R}$ we have that there exists $n \in \mathbb{N}$ such that $x < n$, which means that n is an upper bound of A in $\mathbb{N}$. By Exercise 5 we get that A has a maximum and thus we conclude the desired.

[Reminder: Archimedean property: for all $a, b > 0 \in \mathbb{R}$ there exists $n \in \mathbb{N}$ such that $a < nb$.]

ADVANCED MATHEMATICS 2023-2024
EXERCISES – WEEK 2

* Exercise 1

Prove that a convergent sequence is bounded.

Answer of exercise 1

Let $(a_n)_{n\in\mathbb{N}}$ be a convergent sequence of real numbers and suppose it converges to $l \in \mathbb{R}$. We show that it is bounded. For $\epsilon = 1$ we have that there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$|a_n - l| < 1.$$

This together with the triangle inequality gives us that for all $n \geq N$

$$|a_n| = |a_n - l + l| \leq |a_n - l| + |l| < 1 + |l|.$$

We have found a bound for all a_n with $n \geq N$. To bound all of them consider $M := \max\{1 + |l|, |a_1|, \dots, |a_N|\}$. It is easy to see that for all $n \in \mathbb{N}$

$$|a_n| \leq M.$$

* Exercise 2

Using the "ϵ-definition" of convergence, prove that the sequence $(\frac{1}{n})_{n>1}$ converges to 0.

Take $\epsilon > 0$. Using the Archimedean property of $\mathbb{R}$ for ϵ and 1 we get that there exists $N \in \mathbb{N}$ such that

$$1 < \epsilon \cdot N \Rightarrow \frac{1}{N} < \epsilon.$$

Now, for all $n \geq N$ we get that

$$\left|\frac{1}{n}\right| = \frac{1}{n} \leq \frac{1}{N} < \epsilon.$$

Answer of exercise 2

* Exercise 3

In the lecture, we have seen that given two convergent sequences (u_n) and (v_n), if $u_n \leq v_n$ for all n, then $\lim_{n\to+\infty} u_n \leq \lim_{n\to+\infty} v_n$. What can we say if we instead suppose that for all n, we have

$$u_n < v_n\,?$$

Justify your answer.

Answer of exercise 3

It does not hold that if for all n, $u_n < v_n$ then $\lim_{n\to\infty} u_n < \lim_{n\to\infty} v_n$. Take for example, $\left(\frac{1}{n^2}\right)_{n\in\mathbb{N}}$ and $\left(\frac{1}{n}\right)_{n\in\mathbb{N}}$. They both converge to 0.

** Exercise 4

Let (u_n) be a convergent real sequence with limit $\ell > 0$. Prove that there exists an $N \in \mathbb{N}$ such that for all $n \geq N$, $u_n > 0$.

Answer of exercise 4

Take $\epsilon = \frac{l}{2}$. Then by the definition of convergence we have that there exists $N \in \mathbb{N}$ such that for all $n \geq N$

$$|u_n - l| < \frac{l}{2} \Rightarrow 0 < \frac{l}{2} < u_n < \frac{3l}{2}.$$

** Exercise 5

Let (a_n) and (b_n) be two bounded sequences in $\mathbb{R}$. Prove that

$$\limsup_{n\to+\infty}(a_n + b_n) \leq \limsup_{n\to+\infty} a_n + \limsup_{n\to+\infty} b_n.$$

Can this inequality be promoted to an equality? (Justify with an example.)

Answer of exercise 5

Let $n \in \mathbb{N}$. By definition of supremum, for all $k \geq n$, we have

$$a_k \leq \sup\{a_m \mid m \geq n\}$$

and

$$b_k \leq \sup\{b_m \mid m \geq n\}.$$

It follows that for all $k \geq n$, we have

$$a_k + b_k \leq \sup\{a_m \mid m \geq n\} + \sup\{b_m \mid m \geq n\}.$$

This means that $\sup\{a_m \mid m \geq n\} + \sup\{b_m \mid m \geq n\}$ is an upper bound of the set $\{a_k + b_k \mid k \geq n\}$. Hence, for all $n \in \mathbb{N}$, we have

$$\sup\{a_k + b_k \mid k \geq n\} \leq \sup\{a_m \mid m \geq n\} + \sup\{b_m \mid m \geq n\}$$

(because sup is the least upper bound). Taking the limit of this last equality proves exactly that

$$\limsup_{n\to+\infty}(a_n + b_n) \leq \limsup_{n\to+\infty} a_n + \limsup_{n\to+\infty} b_n.$$

We cannot promote this to an equality. For example, take $a_n = (-1)^n$ and $b_n = (-1)^{n+1}$. Then $a_n + b_n = 0$, so $\limsup(a_n + b_n) = 0$, but $\limsup a_n = 1$ and $\limsup b_n = 1$.

*** Exercise 6

Let (u_n) be a bounded real sequence. Suppose that $m = \sup\{u_n \mid n \geq 0\}$ is not attained (i.e. there is no n such that $u_n = m$). Prove that

$$\limsup_{n\to+\infty} u_n = m.$$

Answer of exercise 6

Let us prove by induction on n that for all $n \geq 0$, we have

$$\sup\{u_k \mid k \geq n\} = m,$$

from which the result follows immediatly.

Base case: for $n = 0$, this is exactly our hypothesis.

Inductive case: suppose that the property is true for a fixed $n \in \mathbb{N}$. Notice that we have

$$\{u_k \mid k \geq n\} = \{u_n\} \cup \{u_k \mid k \geq n+1\}.$$

In particular, we have

$$\sup\{u_k \mid k \geq n\} = \max(u_n, \sup\{u_k \mid k \geq n+1\})$$

(we use here the fact that $\sup A \cup B = \max(\sup A, \sup B)$, which is left as an easy exercise).

Since we assumed that $u_n \neq m$, we have that

$$m = \sup\{u_k \mid k \geq n\} = \sup\{u_k \mid k \geq n+1\}.$$

ADVANCED MATHEMATICS 2023-2024
EXERCISES – WEEK 3

* Exercise 1

Show that if $\sum a_n$ converges, then $\lim a_n = 0$.

Answer of exercise 1

Suppose $(S_n)_{n\in\mathbb{N}}$ converges to $s \in \mathbb{R}$. Take $\epsilon > 0$. By the convergence of $(S_n)_{n\in\mathbb{N}}$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$, $|S_n - s| < \frac{\epsilon}{2}$. For $N' = N + 1$ we have that for all $n \geq N'$, $n \geq N$ and $n - 1 \geq N$. Thus for all $n \geq N'$ we have

$$|S_n - s| < \frac{\epsilon}{2} \text{ and } |S_{n-1} - s| < \frac{\epsilon}{2}.$$

We therefore get that

$$\begin{aligned} |a_n| = |S_n - S_{n-1}| =& |S_n - s + s - S_{n-1}| \\ & \leq |S_n - s| + |S_{n-1} - s| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon. \end{aligned}$$

* Exercise 2

Show that a sequence (a_n) converges if and only if the series $\sum(a_{n+1} - a_n)$ converges.

Answer of exercise 2

First, observe that the partial n-sum of our series is

$$S_n = (a_1 - a_0) + (a_2 - a_1) + \cdots + (a_n - a_{n-1}) + (a_{n+1} - a_n) = a_{n+1} - a_0.$$

Now, the series $\sum_{n\in\mathbb{N}}(a_{n+1} - a_n)$ converges iff $(S_n)_{n\in\mathbb{N}}$ converges iff there exists a real number $s \in \mathbb{R}$ such that

$$\lim_{n\to\infty} (a_{n+1} - a_0) = s.$$

This is equivalent to: there exists $s \in \mathbb{R}$ such that

$$\lim_{n\to\infty} a_n = s + a_0$$

which holds iff $(a_n)_{n\in\mathbb{N}}$ converges.

** Exercise 3

Determine the convergence or divergence of the series $\sum a_n$ when

(a) $a_n = \sqrt{n+1} - \sqrt{n}$,

(b) $a_n = \dfrac{\sqrt{n+1} - \sqrt{n}}{n}$ (*Hint:* Use Proposition 1.4. of the lecture),

(c) $a_n = \dfrac{1}{1 + x^n}$, where x is a non-negative real number (*Hint:* You can use that if $|x| < 1$, then $\lim_{n\to+\infty} x^n = 0$.)

Answer of exercise 3

(a) Observe that $a_n = b_{n+1} - b_n$ where $b_n = \sqrt{n}$. We see that

$$\sum_{n\in\mathbb{N}} a_n = \sum_{n\in\mathbb{N}} (b_{n+1} - b_n)$$

and by Exercise 2 we have that since $b_n = \sqrt{n}$ diverges, so does $\sum_{n\in\mathbb{N}}(b_{n+1} - b_n)$ and thus our series as well.

(b) We see that

$$a_n = \frac{\sqrt{n+1} - \sqrt{n}}{n} = \frac{\sqrt{n+1} - \sqrt{n}}{n} \cdot \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}}$$
$$= \frac{1}{n\sqrt{n+1} + n\sqrt{n}} \leq \frac{1}{n\sqrt{n}} = \frac{1}{n^{\frac{3}{2}}}.$$

Now by Proposition 1.4 since $\frac{3}{2} > 1$ we have that the series $\sum_{n\geq 1} \frac{1}{n^{\frac{3}{2}}}$ converges. By Proposition 1.2 we get that $\sum_{n\geq 1} a_n$ converges since $a_n \leq \frac{1}{n^{\frac{3}{2}}}$ for all $n \in \mathbb{N}$.

(c) For $|x| < 1$ we have that $\lim_{n\in\mathbb{N}} x^n = 0$ therefore $\lim_{n\in\mathbb{N}} \frac{1}{1+x^n} = 1 \neq 0$. By Exercise 1 we have that $\sum_{n\in\mathbb{N}} a_n$ diverges.

Now for $x = 1$ and with the same reasoning as above, we get that the $\sum_{n\in\mathbb{N}} a_n$ diverges since $\lim_{n\in\mathbb{N}} \frac{1}{1+x^n} = \frac{1}{2} \neq 0$.

For $x > 1$ we have that $\frac{1}{x} < 1$ and by Proposition 1.3 we know that $\sum_{n\in\mathbb{N}} \left(\frac{1}{x}\right)^n$ converges. Moreover we see that for all $n \in \mathbb{N}$

$$a_n = \frac{1}{1+x^n} \leq \frac{1}{x^n}$$

and thus by Proposition 1.2 we get that $\sum_{n\in\mathbb{N}} a_n$ converges.

** Exercise 4

Use the root test and the ratio tests on the series

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{2^3} + \frac{1}{3^3} + \cdots.$$

Which one allows to conclude?

Answer of exercise 4

The series is $\sum_{n\geq 2} a_n$ with

$$a_{2p} = \frac{1}{2^p} \text{ for } p \geq 1$$

and

$$a_{2p+1} = \frac{1}{3^p} \text{ for } p \geq 1.$$

Let us start with the ratio test. We have

$$\frac{a_{2p+1}}{a_{2p}} = \left(\frac{2}{3}\right)^p$$

and

$$\frac{a_{2p+2}}{a_{2p+1}} = \frac{1}{2} \cdot \left(\frac{3}{2}\right)^p.$$

In other words, the ratio sequence is

$$\frac{2}{3}, \frac{3}{4}, \frac{4}{9}, \frac{9}{8}, \frac{8}{27}, \frac{27}{16}, \dots$$

Since the sequence $\frac{1}{2}\left(\frac{3}{2}\right)^p$ is not bounded above, then so is $(\frac{a_{n+1}}{a_n})$ and in particular we have

$$\limsup_{n\to+\infty} \frac{a_{n+1}}{a_n} = +\infty,$$

and thus we cannot conclude using the ratio test.

For the root test, we have

$$\sqrt[2p]{a_{2p}} = \sqrt[2p]{\frac{1}{2^p}} = \frac{1}{\sqrt{2}},$$

and

$$\sqrt[2p+1]{a_{2p+1}} = \sqrt[2p+1]{\frac{1}{3^p}} = \frac{1}{3^{\frac{p}{2p+1}}}.$$

Notice that $\frac{p}{2p+1} \le \frac{p}{2p} = \frac{1}{2}$, so $\frac{1}{3^{\frac{p}{2p+1}}} \le \frac{1}{\sqrt{3}} < \frac{1}{\sqrt{2}}$, from which we deduce that

$$\limsup_{n\to+\infty} \sqrt[n]{a_n} = \frac{1}{\sqrt{2}}.$$

This proves that the series is convergent.

* Exercise 5

(1) Show that the series $\sum \frac{1}{n!}$ converges.

(2) Show that the series $\sum(\sqrt[n]{n} - 1)^n$ converges. (*Hint:* Recall that $\lim_{n\to+\infty} \sqrt[n]{n} = 1$)).

Answer of exercise 5

(1) It suffices to apply the ratio test:

$$\lim_{n\to+\infty} \frac{n!}{(n+1)!} = \lim_{n\to+\infty} \frac{1}{n+1} = 0,$$

which proves the convergence.

(2) It suffices to apply the root test:

$$\lim_{n\to+\infty} \sqrt[n]{n} - 1 = 0,$$

which proves the convergence.

** Exercise 6

Let (a_n) be a bounded real sequence and $\alpha = \limsup_{n\to+\infty} a_n$. Show that for every $\beta > \alpha$, there exists an $N \in \mathbb{N}$ such that $a_n < \beta$ for all $n \ge N$.

Answer of exercise 6

Let $\epsilon = \beta - \alpha$. By definition of lim sup, there exists an $N \in \mathbb{N}$ such that for all $n \ge N$

$$\alpha - \epsilon < \sup\{a_k \mid k \ge n\} < \alpha + \epsilon = \beta.$$

In particular, for $n = N$, we have

$$\sup\{a_k \mid k \ge N\} < \beta,$$

which means that for all $k \geq N$,

$$a_k < \beta.$$

ADVANCED MATHEMATICS 2023-2024
EXERCISES – WEEK 4

* Exercise 1

Show that a Cauchy sequence in $\mathbb{R}$ is bounded.

Answer of exercise 1

Let $(x_n)_{n\in\mathbb{N}}$ is a real Cauchy sequence. Since $(x_n)_{n\in\mathbb{N}}$ is Cauchy we have that for $\epsilon = 1$, there exists $N \in \mathbb{N}$ such that for all $m, n \geq N$

$$|x_m - x_n| < 1.$$

Thus for $n = N + 1$ we have that for all $m \geq N$

$$|x_m - x_{N+1}| < 1$$

which gives us that for all $m \geq N$

$$|x_m| = |x_m + x_{N+1} - x_{N+1}| \leq |x_m - x_{N+1}| + |x_{N+1}| < 1 + |x_{N+1}|.$$

For $m < N$ we have that

$$|x_m| \leq \max\{|x_1|, \ldots, |x_{N-1}|\}.$$

Therefore, by choosing the maximum of the above two bounds we have that our sequence is bounded, i.e. for all $m \in \mathbb{N}$

$$|x_m| \leq \max\{\max\{|x_1|, \ldots, |x_{N-1}|\}, 1 + |x_{N+1}|\}.$$

* Exercise 2

Show that the sum of two Cauchy sequences is again a Cauchy sequence.

Answer of exercise 2

Let $(a_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}}$ be two Cauchy sequences. Let $\epsilon > 0$. There exist $N_1, N_2 \in \mathbb{N}$ such that for all $m, n \geq N_i$, $i \in \{1, 2\}$ we have

$$|a_m - a_n| < \frac{\epsilon}{2} \text{ and } |b_m - b_n| < \frac{\epsilon}{2}.$$

So for all $m, n \geq \max\{N_1, N_2\}$ we have

$$\begin{aligned}|(a_m + b_m) - (a_n + b_n)| &= |a_m - a_n + b_m - b_n| \\ &\leq |a_m - a_n| + |b_m - b_n| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

* Exercise 3

Using the Cauchy criterion for series, give another proof that if a real series $\sum a_n$ converges absolutely, then it also converges.

Answer of exercise 3

Suppose that $\sum a_n$ converges absolutely. By definition this means that the series $\sum |a_n|$ is convergent. Using the Cauchy criterion, this means that for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $m \geq n \geq N$, then

$$\left| \sum_{k=n}^{m} |a_k| \right| = \sum_{k=n}^{m} |a_k| < \epsilon.$$

Using the triangular inequality, we have that

$$\left| \sum_{k=n}^{m} a_k \right| \leq \sum_{k=n}^{m} |a_k| < \epsilon.$$

Using the Cauchy criterion again, this proves that the series $\sum a_n$ is convergent.

** Exercise 4

Let $p \geq 0$ be a real number. Let $(a_n)_{n\geq 1}$ be the real sequence given by $a_1 = 1$ and $a_{n+1} = a_n + \frac{1}{n^p}$ for $n \geq 1$. For which values of p is this sequence a Cauchy sequence.

Answer of exercise 4

Because we are in $\mathbb{R}$, a sequence as a Cauchy sequence if and only if it is convergent. By an immediate induction, we have that for all $n \geq 1$,

$$a_n = \sum_{k=1}^{n} \frac{1}{k^p}.$$

Hence, the convergence of the sequence $(a_n)_{n\geq 1}$ means exactly the same thing as the convergence of the series $\sum_{n\geq 1} \frac{1}{n^p}$, which is exactly when $p > 1$ as we have seen in the lecture.

** Exercise 5

Let $x \in \mathbb{R}$ and $y \in \mathbb{R}$. Determine for each of the following case, whether it is a metric or not:

$d_1(x,y) = (x-y)^2$,

$d_2(x,y) = \sqrt{|x-y|}$, (*Hint:* you can use that $\sqrt{a+b} \leq \sqrt{a} + \sqrt{b}$),

$d_3(x,y) = |x^2 - y^2|$,

$d_4(x,y) = |x - 2y|$,

$d_5(x,y) = \dfrac{|x-y|}{1+|x-y|}$ (*Hint:* show that if $a \geq 0$ and $b \geq 0$, then $\dfrac{a+b}{1+a+b} \leq \dfrac{a}{1+a} + \dfrac{b}{1+b}$).

Answer of exercise 5

- d_1 is not a metric as it doesn't satisfy the triangle inequality. For example, if we take $x = 0$, $y = 2$ and $z = 1$, we have

  $$d_1(x,y) = 2^2 = 4 \text{ and } d_1(x,z) + d_1(z,y) = 1^2 + 1^2 = 2.$$

- d_2 is a metric:
  - (Symmetry) $|x-y| = |y-x|$ and so $\sqrt{|x-y|} = \sqrt{|y-x|}$,

- (Separation) if $\sqrt{|x-y|} = 0$, then $|x-y| = 0$, and then $x = y$,
- (Triangle inequality) Recall that we have
$$|x-y| \leq |x-z| + |z-y|,$$
as already seen in the lecture. Since $\sqrt{}$ is non-decreasing, we have
$$\sqrt{|x-y|} \leq \sqrt{|x-y| + |z-y|},$$
and using the hint given, we obtain
$$\sqrt{|x-y|} \leq \sqrt{|x-y|} + \sqrt{|z-y|}.$$

- d_3 is not a metric as it does not satisfy the separation axiom. For example, for $x = 1$ and $y = -1$, we have $d_3(x, y) = 0$.
- d_4 is not a metric as it is not symmetric.
- d_5 is a metric:
  - (Symmetry) From the fact that $|x-y| = |y-x|$, it follows that $d_5(x, y) = d_5(y, x)$.
  - (Separation) If $\frac{|x-y|}{1+|x-y|} = 0$, then $|x-y| = 0$, and then $x = y$.
  - (Triangle inequality) Let's start by proving the inequality given as a hint. For $a, b \in \mathbb{R}_{\geq 0}$, we have
$$\frac{a}{1+a+b} \leq \frac{a}{1+a} \text{ and } \frac{b}{1+a+b} \leq \frac{b}{1+b},$$
and it follows that
$$\frac{a+b}{1+a+b} = \frac{a}{1+a+b} + \frac{b}{1+a+b} \leq \frac{a}{1+a} + \frac{b}{1+b}.$$
Now, let's prove the triangle inequality. First notice that the function $x \mapsto \frac{x}{1+x}$ is non-decreasing (which can be seen by computing the derivative). In particular, since for all $x, y, z \in \mathbb{R}$, we have
$$|x-y| \leq |x-z| + |z-y|,$$
it follows that
$$\frac{|x-y|}{1+|x-y|} \leq \frac{|x-z| + |z-y|}{1+|x-z|+|z-y|} \leq \frac{|x-z|}{1+|x-z|} + \frac{|z-y|}{1+|z-y|},$$
where the second inequality is the one given as a hint.

ADVANCED MATHEMATICS 2023-2024
EXERCISES – WEEK 5

** Exercise 1

Finish the proof of Proposition 1.3 of the lecture.

Answer of exercise 1

Suppose that $(\mathbf{x}_n)$ is convergent to a limit $\ell = (\ell_1, \cdots, \ell_k)$ in $\mathbb{R}^k$. Let $\epsilon > 0$, we know that there exists $N \in \mathbb{N}$ such that if $n \geq N$, then

$$\|\mathbf{x} - \ell\| < \epsilon.$$

By the first inequality given in the lecture, we thus have that for $1 \leq i \leq k$, we have

$$|x_{i,n} - \ell_i| < \epsilon$$

when $n \geq N$. This means exactly that $(x_{i,n})_n$ converges to ℓ_i for each $1 \leq i \leq k$.

Conversely, suppose that each sequence $(x_{i,n})_n$ is convergent to ℓ_i. Let $\epsilon > 0$, we know that for each $1 \leq i \leq k$, there exists an N_i such that if $n \geq N_i$, we have

$$|x_{i,n} - \ell_i| < \epsilon/k.$$

In particular, if we set $N = \max\{N_1, N_2, \cdots, N_k\}$, then if $n \geq N$, then

$$\sum_{i=1}^{k} |x_{i,n} - \ell_i| < \epsilon.$$

It follows then from the second inequality given in the lecture that if $n \geq N$, then

$$\|\mathbf{x}_n - \ell\| < \epsilon,$$

with $\ell = (\ell_1, \cdots, \ell_k)$.

* Exercise 2

Let (X, d) be a metric space and $E \subseteq X$. Show that an element $x \in X$ is an accumulation point of E if and only if it is an adherent point of E which is not isolated.

Answer of exercise 2

The fact that any accumulation point is an adherent point is immediate, because if $E \cap (N_r(x) - \{x\})$ is non-empty, then so is $E \cap N_r(x)$. Moreover, and accumulation point is not isolated, because if $E \cap (N_r(x) - \{x\})$ non-empty for all $r > 0$, then for all $r > 0$, $E \cap N_r(x)$ contains at least another point than x.

Conversely, suppose that x is an adherent point which is not isolated. Let $r > 0$, and consider the subset

$$E \cap (N_r(x) - \{x\}).$$

Since x is an adherent, the subset $E \cap N_r(x)$ is non-empty and necessarily, there exists another element than x in $E \cap N_r(x)$, otherwise this would mean that x is isolated.

* Exercise 3

In each of the following case, determine the isolated points, the accumulation points, the adherence points and the interior points of the subset $E \subseteq \mathbb{R}$:

(1) $E = [0,1] \cup \{2\}$,
(2) $E = [0,1) \cup (1,2]$,
(3) $E = \left\{ \frac{1}{n} \,|\, n \in \mathbb{N} - \{0\} \right\}$.

Answer of exercise 3

(1) isolated points: $\{2\}$, accumulation points: $[0,1]$, adherent points: $[0,1] \cup \{2\}$, interior points: $(0,1)$.
(2) isolated points: none, accumulation points: $[0,2]$, adherent points: $[0,2]$, interior points: $(0,1) \cup (1,2)$.
(3) isolated points: $\left\{ \frac{1}{n}, n \in \mathbb{N} \right\}$, accumulation points: $\{0\}$, adherent points: $\left\{ \frac{1}{n} \,|\, n \in \mathbb{N} - \{0\} \right\}$, interior points: none.

[Note: You need to be able to verify why all the above hold.]

* Exercise 4

Consider the following sequence $\left((-\frac{1}{n}, \frac{1}{n})\right)_{n \geq 1}$ of subsets of $\mathbb{R}$. What is the intersection

$$\bigcap_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n})?$$

Deduce that an (infinite) intersection of open sets is not necessarily an open set.

Answer of exercise 4

We have that

$$\bigcap_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n}) = \{0\}.$$

It is clear that $0 \in (-\frac{1}{n}, \frac{1}{n})$ for all $n \in \mathbb{N}$.

Now, if we take $x \in \bigcap_{n=1}^{\infty} (-\frac{1}{n}, \frac{1}{n})$ with $x \neq 0$, then we have that for all $n \in \mathbb{N}$,

$$-\frac{1}{n} \leq x \leq \frac{1}{n}$$

which means that for all $n \in \mathbb{N}$,

$$0 < |x| \leq \frac{1}{n},$$

holds. By applying the Archimedean Property we have that there exists some $n \in \mathbb{N}$ such that

$$\frac{1}{n} < |x|,$$

which is a contradiction.

* Exercise 5

Let (X, d) be a metric space. Show that any open neighborhood is open in X, that is, for $x \in X$ and $r > 0$, the subset $N_r(x)$ is open in X.

Answer of exercise 5

Let $y \in N_r(x)$. We need to show that there exists an $r' > 0$ such that $N_{r'}(y) \subseteq N_r(x)$. For that, let's take $r' = r - d(x, y)$ and notice that for any $z \in X$ we have

$$d(x, z) \leq d(x, y) + d(y, z),$$

hence, if $d(y, z) < r' = r - d(x, y)$, we have

$$d(x, z) < r,$$

which proves that $N_{r'}(y) \subseteq N_r(x)$.

* Exercise 6

Consider the following subsets of $\mathbb{R}^2$, which ones are open? closed?

(1) The sets of points $\mathbf{x}$ such that $\|\mathbf{x}\| \leq 1$.
(2) The sets of points $\mathbf{x}$ such that $\|\mathbf{x}\| < 1$.
(3) The sets of points of $\mathbb{R}^2$ with integers coordinates.
(4) The sets of points $\mathbf{x} = (x, y)$ such that $y > 0$.

Answer of exercise 6

(1) This set is closed. There are many ways to prove that, here is one of them. By Proposition 2.6 of the lecture notes, it suffices to show that for any convergent sequence $(\mathbf{x}_n)$ in $\mathbb{R}^2$ such that $\|\mathbf{x_n}\| \leq 1$ for all n, then $\|\lim_{n\to+\infty} \mathbf{x}_n\| \leq 1$. But the convergence of $(\mathbf{x_n})$ implies the convergence of the real sequence $(\|\mathbf{x_n}\|)$ to $\|\lim_{n\to+\infty} \mathbf{x}_n\|$ (by the inverse triangle inequality $|\,\|\mathbf{x}\| - \|\mathbf{y}\|\,| < \|\mathbf{x} - \mathbf{y}\|$), and the result follows from the usual results on real sequences.
(2) This subset is nothing than $N_1(\mathbf{0})$, hence we have already proven in Exercise 5 that it is open.
(3) This subset is closed. To see this, let's prove that its complementary is open. Let (x, y) be a point in $\mathbb{R}^2$ such that neither x nor y is an integer. We denote by $E(x)$ and $E(y)$ the integral parts of x and y. Now, if we take $r := \min\{E(x), E(x) + 1, E(y), E(y) + 1\}$, we have that $r > 0$ (because neither x nor y is an integer), and it is easy to see that $N_r((x, y)) \subset E^c$, where E is the subset of $\mathbb{R}^2$ of points with integer coordinates.
(4) This set is open. There are many ways to provat that, for example let us prove that its complementary is closed. Let $((x_n, y_n))$ be a convergent sequence to (ℓ_x, ℓ_y) and such that $y_n \leq 0$. Then, by the usual properties of real sequences, $\ell_y = \lim_{n\to+\infty} y_n \leq 0$, which proves exactly what we want.

* Exercise 7

Prove Corollary 2.13 of the lecture.

Answer of exercise 7

We make use of the Proposition 2.12 of the lecture notes. First, suppose that $E \subseteq Y$ is open relative to Y. Then we have that there exists an open $U \subseteq X$ such that

$$E = U \cap Y.$$

Since E is a binary intersection of two open sets in X, it is open in X.

For the other direction, let $E \subseteq Y$ be an open set in X. Then

$$E = E \cap Y,$$

which means that E is open relative to Y.

ADVANCED MATHEMATICS 2023-2024
EXERCISES – WEEK 6

* Exercise 1

Let A, B be compact subsets of a metric space (X, d). Show that $A \cup B$ and $A \cap B$ are also compact subsets.

Answer of exercise 1

Let $C \subseteq A \cup B$ infinite. We can see that $C = C_1 \cup C_2$, where $C_1 \subseteq A$ and $C_2 \subseteq B$. We can take now the C_i, $i \in \{1, 2\}$ with the largest cardinality and this should be infinite. Say that this is C_1. Then by compactness of A we have that C_1 has a limit point in A. This limit point is also a limit point of C (why?) and we have the desired.

Let $C \subseteq A \cap B$ infinite. Then, $C \subseteq A$ and, since A is compact, there exists a limit point $x \in A$ of C. This means that for every $n \in \mathbb{N} \setminus \{0\}$ there exists $x_n \in C$ such that $d(x, x_n) < \frac{1}{n}$. Observe that $x_n \to x$. Consider now the set $C' = \{x_n : n \in \mathbb{N} \setminus \{0\}\}$. If C' is finite then this means that the sequence is eventually constant and so there exists $n \in \mathbb{N}$ such that $x = x_n \in C' \subseteq C \subseteq A \cap B \subseteq B$. If C' is infinite, then by compactness of B and the fact that $C' \subseteq C \subseteq B$, there exists a limit point $y \in B$ of C'. This again means that for every $k \in \mathbb{N} \setminus \{0\}$ there exists $n_k \in C'$ such that $d(y, x_{n_k}) < \frac{1}{k}$. Therefore, $x_{n_k} \to y$, but since $x_n \to x$, we have also that $x_{n_k} \to x$. By the uniqueness of the limit of a sequence, we deduce that $x = y \in A \cap B$.

[Note that there are other easier ways to solve this, using the characterizations of compactness.]

* Exercise 2

Let (x_n) be a sequence in a metric space (X, d). Show that the following are equivalent:

(1) (x_n) converges to ℓ
(2) every subsequence of (x_n) converges to ℓ,
(3) the subsequences (x_{2n}) and (x_{2n+1}) converges to ℓ.

Answer of exercise 2

(1) $\Rightarrow$ (2). Suppose $(x_n)_n$ converges to l. Let $(x_{n_k})_k$ be a subsequence of $(x_n)_n$. Let $\epsilon > 0$. Then we know since $(x_n)_n$ converges to l that there exists $N \in \mathbb{N}$ such that $\forall n \geq N$ we have:

$$d(x_n, l) < \epsilon.$$

But for all k such that $n_k \geq N$ we now have that:

$$d(x_{n_k}, l) < \epsilon,$$

which concludes that $(x_{n_k})_k$ converges.

(2) $\Rightarrow$ (3). Suppose that every subsequence of $(x_n)_n$ converges to l. Then it is clear that also the subsequences $(x_{2n})_n$ and $(x_{2n+1})_n$ converge to l.

(3) $\Rightarrow$ (1). Suppose that the subsequences $(x_{2n})_n$ and $(x_{2n+1})_n$ converge to l. Let $\epsilon > 0$. We know that these two subsequences converge to l, meaning that there exist $N_1, N_2 \in \mathbb{N}$ such that for all n such that $2n \geq N_1$ and for all n such that $2n+1 \geq N_2$ we have

$$d(x_{2n}, l) < \epsilon \text{ and } d(x_{2n+1}, l) < \epsilon.$$

Now, for $N = \max\{N_1, N_2\}$, we have that for all $n \geq N$, $n = 2k$ or $n = 2k+1$ for some k, meaning that x_n either belongs in the even subsequence or in the odd one. Thus, by the above inequalities, we get that in both cases, for all $n \geq N$

$$d(x_n, l) < \epsilon.$$

Therefore, $(x_n)_n$ converges to l.

** Exercise 3

Determine which of the following subsets of $\mathbb{R}^2$ are compact:

(1) $A = (\mathbb{Q} \cap [0,1]) \times [0,1]$,
(2) $B = \{(x,y) \in \mathbb{R}^2 \mid x = 0\}$,
(3) $C = (\{0\} \cup \{\frac{1}{n} \mid n \in \mathbb{N}_{>0}\}) \times [0,1]$,
(4) $D = \{(\frac{1}{n}, \frac{n-1}{n}) \mid n \in \mathbb{N}_{>0}\}$.

Answer of exercise 3

First, we note that we will make use of the result of Exercise 4 on $\mathbb{R}^2$ and of the following observation:

If $A \times B \subseteq \mathbb{R}^2$ is closed then A and B are also closed. Or in other words, if A and B are not closed then $A \times B$ is also not closed.

(1) A is not compact. It is bounded but is is not closed, since $\mathbb{Q} \cap [0,1]$ is not closed (every irrational is an adherent point of this set but it does not belong in it). [Note here that we made use of the observation above.]
(2) B is not compact, since it is not bounded in the second coordinate.
(3) C is compact. We observe that $C \subseteq [0,1] \times [0,1]$ so it is bounded and that it is also closed. Indeed, let (x,y) be an adherent point of C. If $(x,y) \notin C$ then we can distinguish two cases. Either $(x,y) \in [0,1] \times [0,1]$ or is in the complement of the latter. If it is in the complement then it is simple to find an appropriate radius for a neighborhood around (x,y) so this has no intersection with C, and thus we have a contradiction. Now, if (x,y) is in $[0,1] \times [0,1]$, then the x coordinate is in the interval $(\frac{1}{n}, \frac{1}{n+1})$ for some $n \in \mathbb{N}$. Therefore, by choosing $r < \frac{1}{n(n+1)}$ (the distance of these two vertical lines), we have our desired neighborhood that does not intersect C. Again we arrive at a contradiction. Thus, C is closed.
(4) D is not compact, since it is not closed. It has an accumulation point, namely $(0,1)$, that is not in D. But then $(0,1)$ is obviously also an adherent point which is not in D. Thus, D is not closed.

* Exercise 4

Show that a subset $E \subset \mathbb{R}^k$ is bounded in the sense of Definition 1.11 of the lecture notes if and only if there exists $M_1, M_2, \cdots, M_k$ such that $E \subseteq [-M_1, M_1] \times [-M_2, M_2] \times \cdots \times [-M_k, M_k]$.

Answer of exercise 4

Suppose that $E \subseteq \mathbb{R}^k$, which means that there exists and M such that for all $\mathbf{x}, \mathbf{y} \in E$, we have $\|\mathbf{x} - \mathbf{y}\| \leq M$. If E is empty, then the assertion is trivial (as $\emptyset \subseteq [-M_1, M_1] \times [-M_2, M_2] \times \cdots \times [-M_k, M_k]$ for any choice of $M_1, M_2, \cdots, M_k$). In the case that E is not empty, let $\mathbf{x} \in E$. Notice that for any $\mathbf{y} \in E$, and any $1 \leq i \leq k$ we have

$$|x_i - y_i| \leq \|\mathbf{x} - \mathbf{y}\|,$$

and by the triangle inequality, we have

$$|y_i| \leq |x_i| + |x_i - y_i|.$$

In particular, if we set $M_i = |x_i| + M$, then we have

$$E \subseteq [-M_1, M_1] \times [-M_2, M_2] \times \cdots \times [-M_k, M_k].$$

Conversely, suppose that such $M_1, M_2, \cdots, M_k$ exist. Then, for any $\mathbf{x}$ and $\mathbf{y}$ in E and any $1 \leq i \leq k$, we have

$$|x_i - y_i| \leq 2M_i.$$

Recall now that

$$\|\mathbf{x} - \mathbf{y}\| \leq \sqrt{k} \max_{1 \leq i \leq k} |x_i - y_i|.$$

Thus, if we set $M = \sqrt{k} \max_{1 \leq i \leq k} 2M_i$, we have that for all $\mathbf{x}, \mathbf{y} \in E$,

$$\|\mathbf{x} - \mathbf{y}\| \leq M.$$

NB: You don't have to know metric spaces for the midterm or final exam. The solutions of the following exercices are just given as an indication (and are less detailed that other solutions).

* Exercise 5

Show Proposition 2.4. and 2.5 of the lecture notes.

Answer of exercise 5

- Proposition 2.4: Let $F \subseteq X$ a closed subest of a complete metric space. Let (x_n) be a Cauchy sequence in F. Because X is complete, this sequence admits a limit in X, but because F is closed, this limit is actually in F. Hence, this proves that F is complete.
- Proposition 2.5: Let X be a compact metric space and let (x_n) be a Cauchy sequence in X. By compacity, x_n admits a subsequential limit in X. To conclude let us prove the following general fact.

If a Cauchy sequence (x_n) admits a subsequential limit, then it is convergent. Let (x_{n_k}) be a subsequence which converges to ℓ. Let us show that ℓ is also the limit of (x_n). For any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n, m \geq N$, then

$$d(x_n, x_m) < \epsilon.$$

In particular, for all $n \geq N$ and all $n_k \geq N$, we have

$$d(x_n, x_{n_k}) < \epsilon.$$

By taking the limit when n_k tends to $+\infty$, we obtain

$$d(x_n, \ell) \leq \epsilon$$

(because d is continuous). This proves that ℓ is the limit of (x_n).

ADVANCED MATHEMATICS 2023-2024
EXERCISES – WEEK 7

* Exercise 1

Using the "epsilon-delta" definition of continuity, show that the function $\mathbb{R} \to \mathbb{R}, x \mapsto x^2$ is continuous.

Answer of exercise 1

We need to prove that the function is continuous in all real numbers. Let $x_0 \in \mathbb{R}$ and let $\epsilon > 0$. Choose $\delta = \frac{1}{2}\min\{1, \frac{\epsilon}{1+2|x_0|}\}$. Now, let $x \in \mathbb{R}$. We see that if $|x - x_0| < \delta$ then

$$\begin{aligned}
|f(x) - f(x_0)| &= |x^2 - x_0^2| = |(x - x_0)(x + x_0)| \\
&= |(x - x_0)(x - x_0 + 2x_0)| \\
&\leq |(x - x_0)^2| + |2x_0(x - x_0)| \\
&< \delta^2 + 2|x_0|\delta < 1 \cdot \delta + 2|x_0|\delta \\
&= \delta(1 + 2|x_0|) < \epsilon.
\end{aligned}$$

* Exercise 2

Let $f\colon \mathbb{R} \to \mathbb{R}$ be the function defined as

$$f(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0. \end{cases}$$

Using the "epsilon-delta" definition of continuity, show that the function f is *not* continuous at 0.

Answer of exercise 2

We need to prove that f is not continuous in $x_0 = 0$, so we need to find an $\epsilon > 0$ such that for all $\delta > 0$, there exists $x \in \mathbb{R}$ such that

$$\text{if } |x| < \delta \text{ then } |f(x) - f(0)| \geq \epsilon.$$

Let $\epsilon = 1$. For every $\delta > 0$ let $x = -\frac{\delta}{2}$. Then we see that $|x| < \delta$ holds and that

$$|f(x) - f(0)| \overset{x \leq 0}{=} |0 - 1| = 1 \geq \epsilon.$$

* Exercise 3

Let (X, d) be a metric space. Show that the metric $d\colon X \times X \to \mathbb{R}_{\geq 0}$ is a continuous function.

Answer of exercise 3

To make sense of the statement, we need to equip $X \times X$ with a metric. We are going to use the metric $d_{X \times X}((x, y), (x', y')) = \max\{d(x, x'), d(y, y')\}$, which we have seen in the second homework (applied to the case that $Y =$

X). Notice that a sequence (x_n, y_n) in $X \times X$ converges to (ℓ_x, ℓ_y) if and only if (x_n) converges to ℓ_x and (y_n) converges to ℓ_y in X.

Now let (x_n, y_n) be a sequence converging to (ℓ_x, ℓ_y) in $X \times X$. We need to show that the (real) sequence $(d(x_n, y_n))$ converges to $d(\ell_x, \ell_y)$. Let $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $n \geq N$, then

$$d(x_n, \ell_x) < \epsilon/2,$$

and

$$d(y_n, \ell_y) < \epsilon/2.$$

Now, because of the triangle inequality and symetry, we have that

$$d(x_n, y_n) \leq d(x_n, \ell_x) + d(\ell_x, \ell_y) + d(y_n, \ell_y),$$

from which we deduce that

$$d(x_n, y_n) - d(\ell_x, \ell_y) \leq d(x_n, \ell_x) + d(\ell_y, y_n). \tag{1}$$

Similarly, we have

$$d(\ell_x, \ell_y) \leq d(x_n, \ell_x) + d(x_n, y_n) + d(y_n, \ell_y),$$

from which we deduce that

$$d(\ell_x, \ell_y) - d(x_n, y_n) \leq d(x_n, \ell_x) + d(\ell_y, y_n). \tag{2}$$

Combining (1) et (2) give that

$$|d(x_n, y_n) - d(\ell_x, \ell_y)| \leq d(x_n, \ell_x) + d(\ell_y, y_n).$$

In particular, for $n \geq N$, we have

$$|d(x_n, y_n) - d(\ell_x, \ell_y)| < \epsilon,$$

which means, by definition, that the sequence $(d(x_n, y_n))$ converges to $d(\ell_x, \ell_y)$.

* Exercise 4

Using the properties of continuous functions in relation with open and closed subsets, give a very short proofs that the following subsets of $\mathbb{R}^2$ are open or closed (to be determined in each case):

(1) the subset of $\mathbf{x}$ such that $\|\mathbf{x}\| = 1$,
(2) the subset of $\mathbf{x}$ such that $\|\mathbf{x}\| \leq 1$,
(3) the subset of $\mathbf{x}$ such that $\|\mathbf{x}\| < 1$.

Answer of exercise 4

Consider the function

$$\begin{aligned} \mathbb{R}^2 &\to \mathbb{R} \\ (x, y) &\mapsto \sqrt{x^2 + y^2} = \|(x, y)\|. \end{aligned}$$

This function is continuous, and thus we can deduce that

(1) this subset is closed as it is equal to the pre-image $f^{-1}(\{1\})$ and $\{1\}$ is closed in $\mathbb{R}$,
(2) this subset is closed as it is equal to the pre-image $f^{-1}([0, 1])$ and $[0, 1]$ is closed in $\mathbb{R}$,
(3) this subset is open as it is equal to the pre-image $f^{-1}((-1, 1))$, and $(-1, 1)$ is open in $\mathbb{R}$.

Remark: For the third subset, it is also true that it is equal to $f^{-1}([0,-1))$ (as the function f never takes negative values), but this doesn't give much information because $[0,-1)$ is not open (or closed for that matter). Hence, $(-1,1)$ was conveniently chosen to show that the subset is open.

** Exercise 5

Show that functions $\mathbb{R}^2 \to \mathbb{R}, (x,y) \mapsto \max\{x,y\}$ and $(x,y) \mapsto \min\{x,y\}$ are continuous.

Answer of exercise 5

It suffices to notice that

$$\max\{x,y\} = \frac{x+y+|x-y|}{2}$$

and

$$\min\{x,y\} = \frac{x+y-|x-y|}{2}.$$

Since the absolute value function is continuous, the functions min and max are continuous by composition, sum and multiplication by a scalar of continuous functions.

** Exercise 6

Let $f\colon [0,1] \to [0,1]$ be a continuous function. Show that there exists an $x \in [0,1]$ such that $f(x) = x$. (*Hint:* use the mean-value theorem on a well-chosen function.)

Answer of exercise 6

Let $g\colon [0,1] \to [0,1]$ the function defined by $g(x) = f(x) - x$. This function is continuous and we have

$$g(0) = f(0) - 0 \geq 0$$

and

$$g(1) = f(1) - 1 \leq 1 \text{ (because } f(1) \leq 1 \text{ by hypothesis).}$$

Hence by the mean-value theorem, there exists $0 \leq x_0 \leq 1$ such that $g(x_0) = 0$, which means exactly that $f(x_0) = x_0$.

** Exercise 7

Prove that any polynomial function $f\colon \mathbb{R} \to \mathbb{R}$ of degree 3, $f(x) = ax^3 + bx^2 + cx + d$, with $a \neq 0$, admits at least one real root (that is, there exists $x_0 \in \mathbb{R}$ such that $f(x_0) = 0$).

Answer of exercise 7

Suppose that $a > 0$ (the case $a < 0$ is symetrical). We have

$$\lim_{x \to +\infty} f(x) = +\infty$$

and

$$\lim_{x \to -\infty} f(x) = -\infty.$$

Since f is continuous (as a polynomial function), we can apply the mean-value theorem which shows that there exists an $x_0 \in \mathbb{R}$ such that $f(x_0) = 0$. (*Remark:* To properly apply the mean value theorem as stated in the lecture, it suffices to notice that because the limit at $+\infty$ is $+\infty$ and the limit at

$-\infty$ is $-\infty$, there necessarily exists $x < y$ in $\mathbb{R}$ such that $f(x) < 0$ and $f(y) > 0$.)

ADVANCED MATHEMATICS 2023-2024
EXERCISES – WEEK 9

* Exercise 1

Show Corollary 1.4 of the lecture notes

** Exercise 2

Let $f\colon X \to Y$ be a bijective continuous function (recall that bijective means one-to-one and onto) and let $f^{-1}\colon Y \to X$ be the inverse of f.

Show that if X is compact, then f^{-1} is also continuous.

ADVANCED MATHEMATICS 2023-2024
EXERCISES – WEEK 10

Exercise 1

Find the pointwise limit of the given sequence of functions (f_n) and verify if the convergence is uniform.

(1) $f_n(x) = \dfrac{x}{1+n^2x^2}$, on $I = \mathbb{R}$.

(2) $f_n(x) = \dfrac{nx}{1+n^2x^2}$, on $I = [1,2]$ and $I = [0,1]$.

(3) $f_n(x) = \dfrac{nx}{1+nx^2}$, on $I = [0,1]$ and $I = [\delta, +\infty)$ with $\delta > 0$.

(4) $f_n(x) = \dfrac{1}{1+n^2x^2}$ on $I = [0,1]$, $I = (0,1]$ and $I = [\delta, 1]$ with $0 < \delta < 1$.

(5) $f_n(x) = e^{-|x-n|}$ on $I = \mathbb{R}$.

Exercise 2

Consider the sequence of functions $(f_n)_{n\geq 1}$, where $f_n\colon [0,+\infty)$ is defined by

$$f_n(x) = \frac{1}{\left(x+\frac{1}{n}\right)^n}.$$

(1) For which $x \geq 0$ does the sequence $(f_n(x))$ converge?
(2) Fix a real number $t > 1$. Prove that the sequence (f_n) converges uniformly on $[t, +\infty)$.
(3) Show that the sequence (f_n) does not converge uniformly on $[1, +\infty)$.
(4) Does the sequence (f_n) converge uniformly on $(1, +\infty)$?

Exercise 3

Let (f_n) be a sequence of functions, where $f_n\colon X \to \mathbb{R}$. Show that if $\sum f_n$ converges uniformely on X, then (f_n) converges uniformly to the zero function on X.

Exercise 4

Verify whether the series $\sum_{n=1}^{+\infty} f_n$ converges uniformly on I, for

(1) $f_n(x) = \frac{\sin(nx)}{n^2}$, $I = \mathbb{R}$,
(2) $f_n(x) = e^{nx}$, $I = (-2,-1)$,
(3) $f_n(x) = n(n+1)x^n$, $I = [-1+\delta, 1-\delta]$ with $0 < \delta < 1$.

Exercise 5

Prove Theorem 2.4 of the lecture notes.

ADVANCED MATHEMATICS 2023-2024
EXERCISES – WEEK 11

Exercise 1

Let $f\colon \mathbb{R} \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x}) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Show that the function is derivable on $\mathbb{R}$. Is f' continuous?

Answer of exercise 1

For $x \neq 0$, f is derivable at x (because the product and the composition of derivable functions is derivable), and we have

$$f'(x) = 2x \sin\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right).$$

To show that f is derivable at 0, we compute the following limit

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} x \sin\left(\frac{1}{x}\right) = 0,$$

where the last equality comes from the fact that sin is bounded by 1 and so $|x \sin\left(\frac{1}{x}\right)| \leq |x|$. Hence, we have $f'(0) = 0$.

The function f' is not continuous at 0 (but is continuous everywhere else). To see that, it suffices to notice that the limit

$$\lim_{x\to 0} f'(x) = \lim_{x\to 0} 2x \sin\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right)$$

does not exist. Indeed, since $2x \sin\left(\frac{1}{x}\right)$ converges (to 0) when $x \to 0$, if the previous limit existed, then necessarily $\cos\left(\frac{1}{x}\right)$ would also converges when $x \to 0$, which is not the case.

In particular, we don't have $\lim_{x\to 0} f'(x) = f'(0) = 0$, and thus f' is not continuous at 0.

Exercise 2

Determine for which real numbers a and b the following function f defined on $\mathbb{R}_{\geq 0}$

$$f(x) = \begin{cases} \sqrt{x} & \text{if } 0 \leq x \leq 1, \\ ax^2 + bx + 1 & \text{if } x > 1 \end{cases}$$

is $\mathcal{C}^1$ on $\mathbb{R}_{>0}$ (meaning f is derivable on $\mathbb{R}_{>0}$ and the derivative f' is continuous). Draw a picture of the function.

Answer of exercise 2

Let's start with the continuity. The only problem is for $x = 1$, where we must ensure that

$$\lim_{x\to 1^-} \sqrt{x} = \lim_{x\to 1^+} ax^2 + bx + 1,$$

that is

$$1 = a + b + 1,$$

which gives the condition $a = -b$.

For the derivability, the only problem is also for $x = 1$, and we must ensure that

$$\lim_{x \to 1} \frac{f(x) - f(1)}{x - 1}$$

exists. For that, we can compute the limit for $x \to 1^-$ and for $x \to 1^+$ and ask that they coincide. Because the derivative of $\sqrt{x}$ and $ax^2 + bx + 1$ are derivable at 1, we have

$$\lim_{x \to 1^-} \frac{f(x) - f(1)}{x - 1} = \frac{1}{2\sqrt{1}} = \frac{1}{2}$$

and

$$\lim_{x \to 1^+} \frac{f(x) - f(1)}{x - 1} = 2a + b = a.$$

Hence we obtain the condition $a = \frac{1}{2}$ and $b = -\frac{1}{2}$. Moreover, with these conditions f' is continuous, hence f is $\mathcal{C}^1$, because

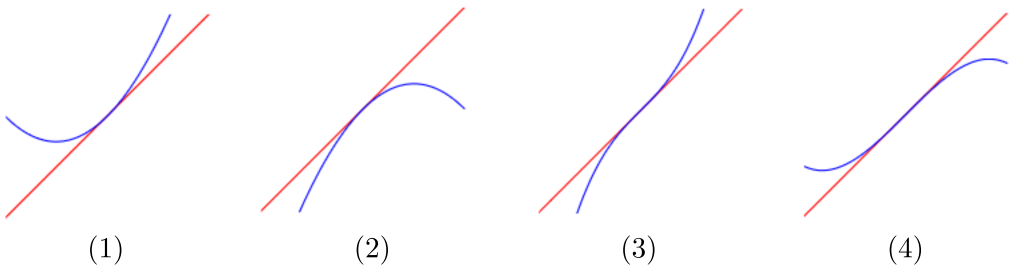
$$\lim_{x \to 1^-} f'(x) = \frac{1}{2} = f'(1) = \lim_{x \to 1^+} f'(x) = 2a + b = \frac{1}{2}.$$

Exercise 3

Let $f\colon I \to \mathbb{R}$ be a function $\mathcal{C}^\infty$ and let $x_0 \in I$. Suppose that there exists a $n > 1$ such that $f^{(n)}(x_0) \neq 0$, and let k be the smallest of such n.

Using the Taylor expansion of f, show the following:

1. if k is even and $f^{(k)}(x_0) > 0$, then f is above its tangent line at x_0,
2. if k is even and $f^{(k)}(x_0) < 0$, then f is below its tangent line at x_0,
3. if k is odd and $f^{(k)}(x_0) > 0$, then f is below its tangent line for $x < x_0$ and above for $x > x_0$,
4. if k is odd and $f^{(k)}(x_0) < 0$, then f is above its tangent line for $x < x_0$ and below for $x > x_0$.



Answer of exercise 3

With the hypotheses, in a neighborhood of x_0, we have

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f^{(k)}(x_0)}{k!}(x - x_0)^k + r(x - x_0),$$

where r is a function such that $\lim_{x \to x_0} \frac{r(x - x_0)}{(x - x_0)^k} = 0$. The tangent line of f at x_0 is the line of equation $y = f(x_0) + f'(x_0)(x - x_0)$, so we want to study the sign of

$$f(x) - (f(x_0) + f'(x_0)(x - x_0))$$

in a neighborhood of x_0. Note that we have

$$\lim_{x \to x_0} k! \frac{f(x) - (f(x_0) + f'(x_0)(x - x_0))}{(x - x_0)^k} = \lim_{x \to x_0} \left(f^{(k)}(x_0) + k! \frac{r(x - x_0)}{(x - x_0)^k} \right) = f^{(k)}(x_0)$$

[Recall that if we have a real function $g \colon I \to \mathbb{R}$ such that if

$$\lim_{x \to x_0} g(x) = \alpha \neq 0,$$

then $g(x)$ is of the sign of α in a neighborhood of x_0 (prove this!).]

In particular, we obtain that, in a neighborhood of x_0,

$$f(x) - (f(x_0) + f'(x_0)(x - x_0))$$

is of the sign of $f^{(k)}(x_0)(x - x_0)^k$.

(1) If k is even and $f^{(k)}(x_0) > 0$, then $f^{(k)}(x_0)(x - x_0)^k$ is positive, hence f is above its tangent line at x_0,
(2) if k is even and $f^{(k)}(x_0) < 0$, then $f^{(k)}(x_0)(x - x_0)^k$ is negative, hence f is below its tangent line at x_0,
(3) if k is odd and $f^{(k)}(x_0) > 0$, then $f^{(k)}(x_0)(x - x_0)^k$ is negative for $x < x_0$ and positive for $x > x_0$, hence f is below its tangent line for $x < x_0$ and above its tangent line for $x > x_0$,
(4) if k is odd and $f^{(k)}(x_0) < 0$, then $f^{(k)}(x_0)(x - x_0)^k$ is positive for $x < x_0$ and negative for $x > x_0$, hence f is above its tangent line for $x < x_0$ and below its tangent line for $x > x_0$.

Exercise 4

Prove the following Taylor expansions at 0:

(1) $e^x = 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + r(x)$,

(2) $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots (-1)^p \frac{x^{2p+1}}{(2p+1)!} + r(x)$,

(3) $\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots + (-1)^p \frac{x^{2p}}{(2p)!} + r(x)$.

Answer of exercise 4

(1) Since the derivative of the exponential function is itself, we obtain

$$\begin{aligned} e^x &= e^0 + e^0 x + e^0 \frac{x^2}{2} + \cdots + e^0 \frac{x^n}{n!} + r(x) \\ &= 1 + x + \frac{x^2}{2} + \cdots + \frac{x^n}{n!} + r(x). \end{aligned}$$

(2) Recall that we have $\sin'(x) = \cos(x)$, and thus $\sin''(x) = -\sin(x)$, $\sin^{(3)}(x) = -\cos(x)$ and $\sin^{(4)}(x) = \sin(x)$. Using that $\sin(0) = 0$ and $\cos(0) = 1$, we obtain the desired formula

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots (-1)^p \frac{x^{2p+1}}{(2p+1)!} + r(x),$$

(3) This is similar to the previous one, using that $\cos'(x) = -\sin(x)$, $\cos''(x) = -\cos(x)$, $\cos^{(3)}(x) = \sin(x)$ and $\cos^{(4)}(x) = \cos(x)$.

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots + (-1)^p \frac{x^{2p}}{(2p)!} + r(x).$$

Exercise 5

Using Taylor expansions, compute the limit of

$$\lim_{x\to 0} \frac{1}{x^2} \left(e^x - \cos x - \sin x\right).$$

Answer of exercise 5

Let's try by using the Taylor expension of degree 2 (this is guided by the fact that we want to have a remainder r such that $\lim_{x\to 0} \frac{r(x)}{x^2} = 0$, to compensate the $1/x^2$ in the limit we want to compute). By the previous exercise, we have

$$e^x = 1 + x + \frac{x^2}{2} + r_1(x), \quad \sin(x) = x + r_2(x) \quad \cos(x) = 1 - \frac{x^2}{2} + r_3(x),$$

with $\lim_{x\to 0} \frac{r_i(x)}{x^2} = 0$ for $i = 1, 2, 3$. Thus we obtain,

$$e^x - \cos x - \sin x = x^2 + r(x),$$

where we set $r(x) = r_1(x) + r_2(x) + r_3(x)$. Hence,

$$\frac{1}{x^2}(e^x - \cos x - \sin x) = 1 + \frac{r(x)}{x^2} \underset{x\to 0}{\longrightarrow} 1.$$

ADVANCED MATHEMATICS 2023-2024
EXERCISES – WEEK 12

* Exercise 1

Let $f\colon \mathbb{R}^m \to \mathbb{R}^n$ be given by

$$f(x) = Ax + b, \quad x \in \mathbb{R}^m,$$

with A an $n \times m$ matrix and $b \in \mathbb{R}^n$. Prove that f is differentiable for $x \in \mathbb{R}^m$ and that $f'(x) = A$.

Answer of exercise 1

We need to prove that

$$\lim_{h\to 0} \frac{\|f(x+h) - f(x) - Ah\|}{\|h\|} = 0.$$

But we have

$$\begin{aligned} f(x+h) - f(x) - Ah &= A(x+h) + b - Ax - b - Ah \\ &= Ax + Ah + b - Ax - b - Ah \\ &= 0. \end{aligned}$$

** Exercise 2

Let $f, g\colon \mathbb{R}^m \to \mathbb{R}^n$ and $\alpha\colon \mathbb{R}^m \to \mathbb{R}$ be differentiable functions. Prove that the functions $F_1, F_2\colon \mathbb{R}^m \to \mathbb{R}^n$ defined below are differentiable and compute their derivatives

$$F_1(x) = f(x) + g(x),$$

$$F_2(x) = \alpha(x) f(x).$$

Answer of exercise 2

- Let's prove that $D_x F_1 = D_x f + D_x g$. This means that we have to prove that

$$\lim_{h\to 0} \frac{\|F_1(x+h) - F_1(x) - D_x f\cdot h - D_x g \cdot h\|}{\|h\|} = 0.$$

We have

$$F_1(x+h) - F_1(x) = f(x+h) + g(x+h) - f(x) - g(x),$$

and so

$$\begin{aligned} \|F_1(x+h) - F_1(x) - D_x f\cdot h - D_x g\cdot h\| &= \|f(x+h) - f(x) - D_x f\cdot h + g(x+h) - g(x) - D_x g\cdot h\| \\ &\le \|f(x+h) - f(x) - D_x f\cdot h\| \\ &\quad + \|g(x+h) - g(x) - D_x g\cdot h\| \end{aligned}$$

where we used the triangle inequality for the last inequality. From this we deduce that

$$\begin{aligned}\lim_{h\to 0}\frac{\|F_1(x+h)-F_1(x)-D_xf\cdot h-D_xg\cdot h\|}{\|h\|} &\leq \lim_{h\to 0}\frac{\|f(x+h)-f(x)-D_xf\cdot h\|}{\|h\|}\\ &\quad+\lim_{h\to 0}\frac{\|g(x+h)-g(x)-D_xg\cdot h\|}{\|h\|}\\ &=0+0.\end{aligned}$$

- This is question is slightly uneasy to answer to "by hand". Let us use the chain rule instead. Let $\varphi\colon \mathbb{R}\times\mathbb{R}^n\to\mathbb{R}^{n+1}$ defined as

  $$\varphi(\lambda,x_1,\cdots,x_m)=(\lambda x_1,\lambda x_2,\cdots,\lambda x_n).$$

  This function is easily seen to be differentiable (prove it!), and the Jacobi matrix is an $n\times(n+1)$ matrix given by

  $$J_{(\lambda,x_1,\cdots,x_n)}\varphi=\begin{pmatrix}x_1 & \lambda & 0 & \cdots & 0\\ 0 & x_2 & \lambda & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & x_n & \lambda\end{pmatrix}.$$

  Now, let $h\colon\mathbb{R}^m\to\mathbb{R}\times\mathbb{R}^n$ defined as $h(x)=(\alpha(x),f(x))$. This function is differentiable, as each components is differentiable by hypothesis, and we have

  $$J_{(x_1,\cdots,x_m)}h=\begin{pmatrix}\frac{\partial\alpha}{\partial x_1} & \cdots & \frac{\partial\alpha}{\partial x_m}\\ \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m}\\ \vdots & & \vdots\\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_m}\end{pmatrix}$$

  Finally, observe that $F_2=\varphi\circ h$, which proves that F_2 is differentiable as a composition of differentiable function and the chain rule gives us that

$$\begin{aligned}J_xF_2&=J_{(\alpha(x),f(x))}\varphi\cdot J_xh\\ &=\begin{pmatrix}f_1(x) & \alpha(x) & 0 & \cdots & 0\\ 0 & f_2(x) & \alpha(x) & \ddots & \vdots\\ \vdots & \ddots & \ddots & \ddots & 0\\ 0 & \cdots & 0 & f_n(x) & \alpha(x)\end{pmatrix}\begin{pmatrix}\frac{\partial\alpha}{\partial x_1} & \cdots & \frac{\partial\alpha}{\partial x_m}\\ \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_m}\\ \vdots & & \vdots\\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_m}\end{pmatrix}\\ &=\begin{pmatrix}f_1(x)J_x\alpha+J_xf_1\alpha(x)\\ \cdots\\ f_n(x)J_x\alpha+J_xf_n\alpha(x)\end{pmatrix}\end{aligned}$$

* Exercise 3

Determine the Jacobi matrix $J_f(x,y)$ for the following functions $f\colon \mathbb{R}^2 \to \mathbb{R}^2$.

$$f(x,y) = \begin{pmatrix} x^2 - y^2 \\ 2xy \end{pmatrix}$$

$$f(x,y) = \begin{pmatrix} e^x \cos y \\ e^x \sin y. \end{pmatrix}$$

Answer of exercise 3

- We have
$$J_x f = \begin{pmatrix} 2x & -2y \\ 2y & 2x \end{pmatrix}.$$
- We have
$$J_x f = \begin{pmatrix} e^x \cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{pmatrix}.$$

*** Exercise 4

Consider the function

$$f(x,y) = \begin{cases} \frac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Compute the partial derivates of f. Is f differentiable in $(0,0)$?

Answer of exercise 4

Let's compute the partial derivatives of f. For $(x,y) \neq (0,0)$, we have

$$\frac{\partial f}{\partial x}(x,y) = \frac{y - 2x}{(x^2+y^2)^2} \text{ and } \frac{\partial f}{\partial y}(x,y) = \frac{x - 2y}{(x^2+y^2)^2}.$$

For $(x,y) = (0,0)$, we must go back to the definition of derivative

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x} = \lim_{x\to 0} \frac{0-0}{x} = 0,$$

and

$$\frac{\partial f}{\partial y}(0,0) = \lim_{y\to 0} \frac{f(0,y) - f(0,0)}{y} = \lim_{y\to 0} \frac{0-0}{y} = 0.$$

The function f is not derivable at $(0,0)$ because it is not even continuous at $(0,0)$. To see that, notice that

$$\lim_{z\to 0} f(z,z) = \lim_{z\to 0} \frac{z^2}{2z^2} = \frac{1}{2} \neq 0.$$

ADVANCED MATHEMATICS 2023-2024
EXERCISES – WEEK 13

Exercise 1

For $\alpha > 0$, let $f_\alpha \colon \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f_\alpha(x,y) = \begin{cases} (x^2+y^2)^\alpha \sin\left(\frac{1}{x^2+y^2}\right) & \text{if } (x,y) \neq (0,0), \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Find (different values) for α such that

(1) the partial derivatives of f_α exist in $(0,0)$;
(2) f_α is differentiable at $(0,0)$;
(3) f_α is $\mathcal{C}_1$ on $\mathbb{R}^2$.

Answer of exercise 1

(1) This means that we have to understand for which values of α the following limits exist

$$\lim_{x\to 0} \frac{f_\alpha(x,0) - f_\alpha(0,0)}{x} = \lim_{x\to 0} x^{2\alpha-1} \sin\left(\frac{1}{x^2}\right)$$

and

$$\lim_{y\to 0} \frac{f_\alpha(y,0) - f_\alpha(0,0)}{y} = \lim_{y\to 0} y^{2\alpha-1} \sin\left(\frac{1}{y^2}\right).$$

That is the case if and only if $\alpha > \frac{1}{2}$. Indeed, it is easy to see that the condition is sufficient because the sin function is bounded and if $\alpha > \frac{1}{2}$, then $\lim_{z\to 0} z^{2\alpha-1} = 0$. To see that this condition is also necessary, notice first that if $\alpha < \frac{1}{2}$, then $\lim_{z\to 0} z^{2\alpha-1} = +\infty$. Then, if we take $z_n = \sqrt{\frac{2}{5n\pi}}$, we have $\sin\left(\frac{1}{z_n^2}\right) = 1$ and $\lim_{n\to+\infty} z_n = 0$. Thus, we have

$$\lim_{n\to+\infty} z_n^{2\alpha-1} \sin\left(\frac{1}{z_n^2}\right) = +\infty,$$

which proves that the limit of $z^{2\alpha-1}\sin\left(\frac{1}{z^2}\right)$ does not exist in 0 (at least the limit cannot be a *real* number, but actually it does not diverge to $+\infty$ either (prove this!)). Finally, in the case $\alpha = \frac{1}{2}$, then $z^{2\alpha-1} = 1$ and $\sin\left(\frac{1}{z^2}\right)$ does not converge when $z \to 0$.

In conclusion, the partial derivatives at $(0,0)$ exist if and only if $\alpha > \frac{1}{2}$ and in this case $\frac{\partial f}{\partial x}(0,0) = \frac{\partial f}{\partial y}(0,0) = 0$.

(2) We know that *if* f is differentiable, then the partial derivatives exist (so we must have $\alpha > \frac{1}{2}$ by the previous question) and the Jacobian of f at $(0,0)$ is given by

$$J_{(0,0)}f = \begin{pmatrix} \frac{\partial f}{\partial x}(0,0) & \frac{\partial f}{\partial y}(0,0) \end{pmatrix} = \begin{pmatrix} 0 & 0 \end{pmatrix}.$$

So, asking that f is differentiable at 0, means that we have to find values of α (with at least $\alpha > \frac{1}{2}$) such that the following limit exists

$$\lim_{(x,y)\to(0,0)} \frac{\left| f_\alpha(x,y) - f_\alpha(0,0) - J_{(0,0)}f \cdot \begin{pmatrix} x \\ y \end{pmatrix} \right|}{\|(x,y)\|} = \lim_{(x,y)\to(0,0)} \frac{|f_\alpha(x,y)|}{\|(x,y)\|}$$
$$= \lim_{(x,y)\to(0,0)} (x^2+y^2)^{\alpha-\frac{1}{2}} \left| \sin\left(\frac{1}{x^2+y^2}\right) \right|.$$

Once again, this happens exactly if and only if $\alpha > \frac{1}{2}$. To see that, observe that $\lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2} = 0$, and so we are reduced to study the existence of the limit when $z \to 0$ of the function $z \mapsto z^{2\alpha-1} \left|\sin\left(\frac{1}{z^2}\right)\right|$, which we have already done in the previous question.

(3) We know from the lecture that a function is $\mathcal{C}^1$ if and only if its partial derivatives exist and are continuous. Outside of $(0,0)$, this will always work for f_α, so the only question is at $(0,0)$. Since a $\mathcal{C}^1$ function is in particular derivable, we know from the previous question that we need at least $\alpha > \frac{1}{2}$. The only thing that we require now is the continuity of the partial derivative at $(0,0)$, that is, we need to have

$$\lim_{(x,y)\to(0,0)} \frac{\partial f_\alpha}{\partial x}(x,y) = \frac{\partial f_\alpha}{\partial x}(0,0)$$

and

$$\lim_{(x,y)\to(0,0)} \frac{\partial f_\alpha}{\partial y}(x,y) = \frac{\partial f_\alpha}{\partial y}(0,0).$$

Let's treat the first one of these equalities, the other one being completely symetrical (as the role of x and y are symetrical in the definition of f_α). For $(x,y) \neq (0,0)$, we compute

$$\frac{\partial f_\alpha}{\partial x}(x,y) = 2x(x^2+y^2)^{\alpha-2} \left(\alpha(x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right) - \cos\left(\frac{1}{x^2+y^2}\right) \right),$$

and we have to find the value of α (with at least $\alpha > \frac{1}{2}$) such the limit of the previous expression when $(x,y) \to (0,0)$ exists and is 0 (because we know that we have $\frac{\partial f_\alpha}{\partial x}(0,0) = 0$).

This happens exactly if and only if $\alpha > \frac{3}{2}$. To see that it is a sufficient condition, notice that

$$|2x(x^2+y^2)^{\alpha-2}| \leq 2\sqrt{x^2+y^2}(x^2+y^2)^{\alpha-2} = 2(x^2+y^2)^{\alpha-\frac{3}{2}}.$$

Since $\lim_{(x,y)\to(0,0)}(x^2+y^2)^{\alpha-\frac{3}{2}} = 0$, by the sandwhich lemma, we also have $\lim_{(x,y)\to(0,0)} 2x(x^2+y^2)^{\alpha-2} = 0$. Since cos is bounded this proves that $2x(x^2+y^2)^{\alpha-2}\cos\left(\frac{1}{x^2+y^2}\right)$ tends to 0 when $(x,y) \to (0,0)$. Moreover, the part with sin in the expression of $\frac{\partial f_\alpha}{\partial x}(x,y)$ also tends to 0 because of the factor (x^2+y^2) in front of it.

To see that the condition $\alpha > \frac{3}{2}$ is also necessary, let us suppose that $\alpha \leq \frac{3}{2}$. Notice that the part with sin will still converges to 0 (for any value of α), so we only need to show that when $\alpha \leq \frac{3}{2}$, the limit

$$\lim_{(x,y)\to(0,0)} 2x(x^2+y^2)^{\alpha-2}\cos\left(\frac{1}{x^2+y^2}\right)$$

does not exist. [Recall that if $\lim_{(x,y)\to(a,b)} f(x,y)$ exists, then we have $\lim_{(x,y)\to(a,b)} f(x,y) = \lim_{x\to a}\lim_{y\to b} f(x,y)$ (prove this!)]. In particular, suppose that the previous limit exist, then we can first take the limit for $y \to 0$ and observe that the limit

$$\lim_{x\to 0} 2x(x^2)^{\alpha-2}\cos\left(\frac{1}{x^2}\right) = \lim_{x\to 0} 2x^{2\alpha-3}\cos\left(\frac{1}{x^2}\right),$$

does not exist when $\alpha \leq \frac{3}{2}$. This proves, by contradiction, that $\lim_{(x,y)\to(0,0)} 2x(x^2+y^2)^{\alpha-2}\cos\left(\frac{1}{x^2+y^2}\right)$ does not exist when $\alpha \leq \frac{3}{2}$.

Exercise 2

Let $f\colon \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$$

Is f a $\mathcal{C}^1$-function on $\mathbb{R}^2$? Is f a $\mathcal{C}^2$-function on $\mathbb{R}^2$? Compute $\frac{\partial^2 f}{\partial x \partial y}(0,0)$ and $\frac{\partial^2 f}{\partial y \partial x}(0,0)$.

Answer of exercise 2

We know from the lecture that a function is $\mathcal{C}^1$ if and only if the partial derivatives exist and are continuous. Outside of $(0,0)$, it is trivially the case, and we only need to see what happens at $(0,0)$. Let us first prove that the partial derivatives exist at $(0,0)$. Observe that

$$\lim_{x\to 0} \frac{f(x,0)-f(0,0)}{x} = \lim_{x\to 0} 0 = 0.$$

Hence, $\frac{\partial f}{\partial x}(0,0) = 0$, and similarly for the other partial derivative. Let us now see that the partial derivatives are continuous, for $(x,y) \neq (0,0)$, we compute that

$$\frac{\partial f}{\partial x}(x,y) = \frac{y\left[(3x^2-y^2)(x^2+y^2) - 2x^2(x^2-y^2)\right]}{(x^2+y^2)^2}.$$

To see that the limit of this function is 0 when $(x,y) \to (0,0)$, observe that

$$\begin{aligned}\frac{\left|y\left[(3x^2-y^2)(x^2+y^2) - 2x^2(x^2-y^2)\right]\right|}{(x^2+y^2)^2} &\leq \frac{|y|\left[(3x^2+y^2)(x^2+y^2) + 2x^2(x^2+y^2)\right]}{(x^2+y^2)^2} \\ &\leq \frac{|y|(5x^2+y^2)}{(x^2+y^2)} \\ &\leq \frac{|y|5(x^2+y^2)}{(x^2+y^2)} = 5|y|.\end{aligned}$$

Thus, by the sandwhich lemma, $\lim_{(x,y)\to(0,0)} \frac{\partial f}{\partial x}(x,y) = 0 = \frac{\partial f}{\partial x}(0,0)$. For the other partial derivative, observe that $f(x,y) = -f(y,x)$, and thus

$$\frac{\partial f}{\partial y}(x,y) = -\frac{\partial f}{\partial x}(y,x),$$

which proves that this partial derivative is also continuous. Hence, f is $\mathcal{C}^1$.

Let us prove that f not $\mathcal{C}^2$. In order to do that, let's compute $\frac{\partial^2 f}{\partial y \partial x}(0,0)$, which is the following limit

$$\lim_{y\to 0} \frac{\frac{\partial f}{\partial x}(0,y) - \frac{\partial f}{\partial x}(0,0)}{y} = \frac{-y^5}{y^5} = -1,$$

and then let's compute $\frac{\partial^2 f}{\partial x \partial y}(0,0)$

$$\begin{aligned}\lim_{x\to 0} \frac{\frac{\partial f}{\partial y}(x,0) - \frac{\partial f}{\partial y}(0,0)}{x} &= \lim_{x\to 0} \frac{-\frac{\partial f}{\partial x}(0,x)}{x} \\ &= -\frac{-x^5}{x^5} = 1.\end{aligned}$$

Therefore, f cannot be $\mathcal{C}^2$, otherwise we would have $\frac{\partial^2 f}{\partial y \partial x}(0,0) = \frac{\partial^2 f}{\partial x \partial y}(0,0)$.

ADVANCED MATHEMATICS 2023-2024
EXERCISES – WEEK 14

** Exercise 1

Use the chain rule to prove that the sum of two differentiable functions is again differentiable, and compute the derivative.

Answer of exercise 1

Let $f, g\colon \mathbb{R}^m \to \mathbb{R}^n$ be two differentiable functions. We would like to show that the function $f + g\colon \mathbb{R}^m \to \mathbb{R}^n$ defined as

$$(f+g)(x) = f(x) + g(x)$$

is differentiable and compute its derivative. For that, consider the function

$$S\colon \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$$
$$(x, y) \mapsto (x + y).$$

Notice that S is a linear map $\mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$, whose matrix is given by

$$M(S) = \begin{pmatrix} 1 & & 0 & 1 & & 0 \\ & \ddots & & & \ddots & \\ 0 & & 1 & 0 & & 1. \end{pmatrix}$$

In particular, S is differentiable (cf. Exercise 1 Week 12) and the Jacobi matrix is given by $M(S)$.

Notice now that $f + g = S \circ (f, g)$, where (f, g) is the function

$$(f, g)\colon \mathbb{R}^m \to \mathbb{R}^n \times \mathbb{R}^n$$
$$x \mapsto (f(x), g(x)),$$

which is differentiable (why?). Hence, $f + g$ is differentiable by composition of differentiable functions. To compute the derivate (or rather the Jacobi matrix) of $f + g$, notice that the Jacobi matrix of (f, g) is given by

$$J_x(f, g) = \begin{pmatrix} J_x f \\ J_x g \end{pmatrix},$$

that is, is obtained by stacking on top of each other the Jacobi matrices of f and g. Hence by the chain rule, we obtain

$$J_x(f + g) = M(S) J_x(f, g) = J_x f + J_x g,$$

or,

$$D_x(f + g) = D_x f + D_x g.$$

** Exercise 2

Let $E \subset \mathbb{R}^n$ be an open set, and $f, g\colon E \to \mathbb{R}$ be two differentiable functions. Use the chain rule to show that the product fg is differentiable and compute the partial derivative $\frac{\partial(fg)}{\partial x_j}$. Give a formula for the derivative $D_x(fg)$ for all $x \in E$.

Answer of exercise 2

The idea is the same as for the previous exercise. Let $\mu\colon \mathbb{R}\times\mathbb{R}\to\mathbb{R}$ be defined as

$$\mu(a,b) = a\times b.$$

This function is differentiable (this can be seen, for example, by the fact that its partial derivatives are $\mathcal{C}^\infty$). Moreover, the Jacobi matrix of μ is given by

$$J_{(a,b)}\mu = \begin{pmatrix} b & a \end{pmatrix}.$$

Notice now that we have $fg = \mu\circ(f,g)$, where (f,g) is the function

$$\begin{aligned}(f,g)\colon E &\to \mathbb{R}\times\mathbb{R}\\ x &\mapsto (f(x),g(x)).\end{aligned}$$

This function is differentiable (why?) and the Jacobi matrix is given by

$$J_x(f,g) = \begin{pmatrix} J_xf \\ J_xg\end{pmatrix}.$$

In particular, fg is differentiable and we have

$$\begin{aligned}J_xfg &= J_{(f(x),g(x))}\mu\ J_{(x)}(f,g)\\ &= g(x)J_{(x)}f + f(x)J_{(x)}g.\end{aligned}$$

This gives

$$D_x(fg) = g(x)D_xf + f(x)D_xg$$

and

$$\frac{\partial(fg)}{\partial x_j}(x) = g(x)\frac{\partial f}{\partial x_j}(x) + f(x)\frac{\partial g}{\partial x_j}(x)$$

* Exercise 3

Let $U = \{(x,y)\in\mathbb{R}^2 | x>0, y>0\}$, $V = \{(x,y)\in\mathbb{R}^2 | y>0\}$, and $\varphi\colon U\to V$ given by $\varphi(x,y) = (x^2-y^2, 2xy)$. Prove that φ is a diffeomorphism.

Answer of exercise 3

An easy computation gives us that the Jacobi matrix of φ is given by

$$J_{(x,y)}\varphi = \begin{pmatrix} 2x & -2y\\ 2y & 2x\end{pmatrix}.$$

Its determinant is then given by

$$\det(J_{(x,y)}\varphi) = 2x^2+2y^2,$$

which is always >0 on U. In particular $J_{(x,y)}$ is invertible on U and by the inverse function theorem, we know that φ is a local diffeomorphism. In order to prove that it is a diffeomorphism $\varphi\colon U\to V$, it suffices to prove that $\varphi(U)=V$ and that it is injective.

Notice first that we have $\varphi(U)\subseteq V$ because if $x>0$ and $y>0$, then $2xy>0$. For the converse, let $b>0$. We need to find $x>0$ and $y>0$ such that $2xy=b$, which is trivial (for example $x=1$ and $y=b$). Hence we have $\varphi(U)=V$.

For the injectivity, let $x,x',y,y'>0$ such that

$$x^2-y^2 = x'^2-y'^2 \text{ and } 2xy = 2x'y'.$$

From the second equation, we deduce that $x = x'\frac{y'}{y}$. Injecting in the first equation, we easily get

$$\frac{x'^2}{y^2}(y'^2 - y^2) = y^2 - y'^2.$$

Since $\frac{x'^2}{y^2} > 0$, we necessarily have $y^2 = y'^2$. Since we suppose $y, y' > 0$, we then have $y = y'$. The equality $x = x'$ follows at once from the condition $xy = x'y'$.

All in all, this proves that φ is injective and since it is a local diffeomorphism, it is thus a diffeomorphism onto its image $\varphi(U) = V$.

** Exercise 4

Let $f\colon \mathbb{R}^2 \to \mathbb{R}^2$, given by $f(x, y) = (e^x \cos y, e^x \sin y)$.

(1) What is the range of f?
(2) Show that the Jacobi matrix of f is invertible at any point of $\mathbb{R}^2$. Thus, by the inverse function theorem, every point of $\mathbb{R}^2$ has a neighborhood in which f is one-to-one. Is f one-to-one on $\mathbb{R}^2$?
(3) Let $a = (0, \pi/3)$, $b = f(a)$ and let g be the inverse of f, defined in a neighborhood of b, such that $g(b) = a$ (whose existence follows from the inverse function theorem). Find an explicit formula for g, compute $D_a f$ and $D_b g$ and verify that

$$D_b(g) = D_a(f)^{-1}.$$

Answer of exercise 4

(1) Note that $(e^x \cos y, e^x \sin y) \neq (0, 0)$ for any $(x, y) \in \mathbb{R}^2$ (because this would force both $\sin y = 0$ and $\cos y = 0$ which is impossible). Conversely let $(a, b) \neq (0, 0)$ a point of $\mathbb{R}^2$. Let r be the distance from $(0, 0)$ to (a, b) (i.e. $r = \sqrt{a^2 + b^2}$) and θ be a value of the angle between the x axis and the half-line determine by $(0, 0)$ and (a, b) (which is well defined since $(a, b) \neq (0, 0)$). Then we have $e^{\ln(r)} \cos\theta = a$ and $e^{\ln(r)} \sin\theta = y$. This proves that the range of f is $\mathbb{R}^2 - \{0\}$.
(2) The Jacobi matrix of f is given by

$$J_{(x,y)}f = \begin{pmatrix} \cos y & -e^x \sin y \\ \sin y & e^x \cos y. \end{pmatrix}$$

We have $\det(J_{(x,y)}f) = 2e^x > 0$, hence $J_{(x,y)}f$ is invertible for every point of $\mathbb{R}^2$. We deduce by the inverse function theorem that f is one-to-one locally around any point of $\mathbb{R}^2$. However, f is not one-to-one on $\mathbb{R}^2$, for $f(x, y) = f(x, y + 2k\pi)$ for any $k \in \mathbb{Z}$.
(3) We have $g(x, y) = (\ln(\sqrt{x^2 + y^2}), \arctan\left(\frac{y}{x}\right))$, where arctan is the inverse of the function $\tan\colon (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$. We compute

$$J_a f = \begin{pmatrix} \cos\left(\frac{\pi}{3}\right) & -\sin\left(\frac{\pi}{3}\right) \\ \sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right) \end{pmatrix},$$

$b = (\cos\left(\frac{\pi}{3}\right), \sin\left(\frac{\pi}{3}\right))$, and

$$J_b g = \begin{pmatrix} \cos\left(\frac{\pi}{3}\right) & \sin\left(\frac{\pi}{3}\right) \\ -\sin\left(\frac{\pi}{3}\right) & \cos\left(\frac{\pi}{3}\right) \end{pmatrix}.$$

A straightforward verification gives that

$$J_a f \cdot J_b g = J_b g \cdot J_a f = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

which shows exactly that $J_b g$ is the inverse of $J_a f$.

** Exercise 5

Let $f\colon \mathbb{R} \to \mathbb{R}$ defined by

$$f(t) = \begin{cases} t + 2t^2 \sin\left(\frac{1}{t}\right) & \text{if } t \neq 0, \\ 0 & \text{if } t = 0. \end{cases}$$

Show that f is derivable and $f'(0) = 1$, but that f is not one-to-one in any neighborhood of 0. What hypothesis of the inverse function theorem is missing?

Answer of exercise 5

We easily compute that

$$\lim_{t\to 0} \frac{t + 2t^2 \sin\left(\frac{1}{t}\right)}{t} = \lim_{t\to 0} 1 + 2t \sin\left(\frac{1}{t}\right) = 1.$$

Hence, f is derivable at 0 (and also elsewhere trivially) and $f'(0) = 1$.

Let us show that for any $\delta > 0$, there exists $t \in (-\delta, \delta)$ such that $t \neq 0$ and $f(t) = 0$, which shows, in particular that f is not one-to-one in any neighborhood of 0. Consider the equation

$$t + 2t^2 \sin\left(\frac{1}{t}\right) = 0$$

$$t(1 + 2t \sin\left(\frac{1}{t}\right)) = 0.$$

Since we want a solution different than 0, we must solve the equation

$$1 + 2t \sin\left(\frac{1}{t}\right) = 0,$$

which is equivalent (assuming $t \neq 0$) to

$$\sin\left(\frac{1}{t}\right) = -\frac{1}{2t}.$$

This equation has an infinite number of solutions, at least one of which is $0 < t < \delta$ (why?), which proves our claim.

The hypothesis missing to apply the inverse function theorem is that f be $\mathcal{C}^1$. Indeed, outside of 0, we have

$$f'(t) = 1 + 4t \sin\left(\frac{1}{t}\right) - 2\cos\left(\frac{1}{t}\right),$$

and because of the cos term, the limit of $f'(t)$ when $t \to 0$ does not exist. In particular f' is not continuous at 0 and hence f is not $\mathcal{C}^1$.