Final Exam: Grondslagen van de Wiskunde

January 29, 2026, 13:30-16:30

THIS EXAM CONSISTS OF 5 EXERCISES; SEE ALSO THE BACK SIDE.
Every exercise is worth 10 points; your grade is the total divided by 5. If an exercise contains several questions, the number of points attributed to each question is indicated.
The exam is **closed book**. I have recalled tomorrow the Tarski–Vaught test, which you can use freely.
Advice: start first with the exercises that you can do, and then think about the rest. In particular, exercise 3 is harder than the rest. Succes!

~~**Tarski–Vaught test:** N is an elementary substructure of M if and only if for every L-formula $\phi(y, \vec{x})$ with $(n+1)$ free variables and every n-tuple $\vec{a}$ of elements of N,~~

~~if $M \models \exists y \phi(y, \vec{a})$ then $N \models \exists \phi(y, \vec{a})$.~~

Tarski–Vaught test: Let $N \subseteq M$ be a substructure. Then, N is an elementary substructure of M if and only if for every L formula of the form $\exists y \phi(y, \vec{a})$, where $\vec{a}$ is a tuple of N,

$$\text{if } M \models \exists y \phi(y, \vec{a}), \text{ there there exists } b \in N \text{ such that } M \models \phi(b, \vec{a}).$$

Exercise 1. A poset $(I, \leq)$ is *directed* if (a) it is non-empty and (b) for all $i, j \in I$, there exists a $k \in I$ such that $i \leq k$ and $j \leq k$. For a language L, we call *directed system of L-structures* a family $(A_i)_{i \in I}$ of L-structures indexed by a directed poset $(I, \leq)$ and such that for $i \leq j$, A_i is a substructure of A_j (and in particular $A_i \subseteq A_j$).

1. (2 pts) Let L be a language and let $(A_i)_{i \in I}$ be a directed system of L-structures. Show that there exists a unique way to make the set $A_\infty := \cup_{i \in I} A_i$ an L-structure such that each A_i is a substructure of A_∞.
2. (3 pts) In the situation of the previous question, show that if we suppose furthermore that for $i \leq j$, A_i is an elementary substructure of A_j, then each A_i is an elementary substructure of A_∞.

We say that a theory T is a $\forall\exists$-theory if the sentences of T are all of the form $\forall \vec{x}\, \exists \vec{y} \phi(\vec{x}, \vec{y})$ where ϕ is *quantifier-free*. We say that a theory T is $\forall\exists$-axiomatizable if there exists a $\forall\exists$-theory T', such that for every sentence ϕ, $T \vdash \phi$ if and only if $T' \vdash \phi$.

3. (3 pts) Show that the models of a $\forall\exists$-theory T are stable by directed union. That is, if we take a union of a directed system as in question (1) such that each A_i is a model of T, then A_∞ is a model of T.
4. (2pts) Show that the theory of posets with a top element (in the language with only one binary relation $\leq$) is not $\forall\exists$-axiomatisable.

Solution:

1. Let us start by the uniqueness. Let F be an n-ary function symbol of L, and let $a_1, \ldots, a_n$ be n elements of A_∞. Using the directedness property, by an immediate induction there exists an $i \in I$, such that $a_1, \ldots, a_n$ are all elements of A_i. Hence, because A_i hast to be a substructure of A_∞, then we have

   $$F^{A_\infty}(a_1, \ldots, a_n) = F^{A_i}(a_1, \ldots, a_n).$$

   This clearly proves that F^{A_∞} (provided this interpretation is possible) is unique. The same argument holds for relation symbols. For constant symbols, this is the same argument, using in addition that I is not empty by definition.

   Now let's prove that such a definition makes sense. That is we have to prove that for $i, j \in I$, $a_1, \ldots, a_n \in A_i \cap A_j$, we have

   $$F^{A_i}(a_1, \ldots, a_n) = F^{A_j}(a_1, \ldots, a_n).$$

   By directedness, there exists a $k \in I$, such that $i \leq k$ and $j \leq k$. Using that A_i and A_j are both substructures of A_k, we have

   $$F^{A_i}(a_1, \ldots, a_n) = F^{A_k}(a_1, \ldots, a_n) = F^{A_j}(a_1, \ldots, a_n).$$

   The same argument holds for constant symbols and for relation symbols.

2. First recall that every formula $\phi(x_1, \dots, x_n)$ with n free variables is equivalent to a formula

$$Q_k y_k \dots Q_1 y_1 \psi(y_1, \dots, y_k, x_1, \dots, x_n),$$

where each symbol Q_i is either $\forall$ or $\exists$, and ψ is a quantifier-free formula with $n+k$ free variables. (See Exercice 62 of the textbook). When $k = 0$, this simply means that ϕ is equivalent to a quantifier-free formula.

Consider the following statement, denoted by H_k, which depends on k:

For all $i \in I$, for all $n \in \mathbb{N}$, for every formula $\psi(\vec{y}, \vec{x})$ with $n+k$ free variables, for every n-tuple $\vec{a}$ of A_i,

$$\text{if } A_\infty \models Q_k y_k \dots Q_1 y_1 \psi(y_1, \dots, y_k, \vec{a}) \text{ then } A_i \models Q_k y_k \dots Q_1 y_1 \psi(y_1, \dots, y_k, \vec{a}).$$

By what was said above, if we can prove H_k for every $k \geq 0$, then this proves that for all $i \in I$, A_i is an elementary substructure of A_∞. We proceed by induction on k. For $k = 0$, this follows immediatly from the fact that A_i is a substructure of A_∞. Suppose now that the statement H_k is true for a fixed k and consider $i \in I$, $n \geq 0$, a quantifier-free formula $\psi(y_{k+1}, y_k, \dots, y_1, \vec{x})$ with $n+k+1$ free variable, and an n-tuple $\vec{a}$ of A_i. Suppose that

$$A_\infty \models Q_{k+1} y_{k+1} \dots Q_1 y_1 \psi(y_{k+1}, y_k, \dots, y_1, \vec{a}),$$

for some sequence of quantifiers $Q_1, Q_2, \dots, Q_{k+1}$. We need to prove that the same sentence holds in A_i. There are two cases to consider:

(a) if $Q_1 = \forall$, then let $b \in A_i$. We have $A_\infty \models Q_k y_k \dots Q_1 y_1 \psi(b, y_k, \dots, y_1, \vec{a})$. Notice now that the order of the argument in ψ has no importance and is purely notational. We can then apply the induction hypothesis H_k with the $(n+1)$-tuple $(b, \vec{a})$, to deduce that $A_i \models Q_k y_k \dots Q_1 y_1 \psi(b, y_k, \dots, y_1, \vec{a})$. Since this is true for every $b \in A_i$, this means exactly that

$$A_i \models \forall y_{k+1} Q_k y_k \dots Q_1 y_1 \psi(y_{k+1}, y_k, \dots, y_1, \vec{a}).$$

(b) if $Q_1 = \exists$, then there exists some $b \in A_\infty$ such that $A_\infty \models Q_k y_k \dots Q_1 y_1 \psi(b, y_k, \dots, y_1, \vec{a})$. By definition of A_∞, there exists $j \in J$ such that $b \in A_j$. By directedness, there exists $l \in I$ such that $i, j \leq l$. In particular, we have $b \in A_l$ and $\vec{a} \in A_l^n$. We can now apply the induction hypothesis H_k for l and the $(n+1)$-tuple $(b, \vec{a})$ to get that

$$A_l \models Q_k y_k \dots Q_1 y_1 \psi(b, y_k, \dots, y_1, \vec{a}).$$

Since, by hypothesis, A_i is an elementary substructure of A_l, we also get

$$A_i \models Q_k y_k \dots Q_1 y_1 \psi(b, y_k, \dots, y_1, \vec{a}),$$

which means exactly that

$$A_i \models \exists y_{k+1} Q_k y_k \dots Q_1 y_1 \psi(y_{k+1}, y_k, \dots, y_1, \vec{a}).$$

3. Consider a sentence of the form $\forall \vec{x} \exists \vec{y} \phi(\vec{x}, \vec{y})$, where $\vec{x}$ is an n-tuple of variables, and $\vec{y}$ is a k-tuple of variables, which holds in every A_i. Now let $\vec{a}$ be an n-tuple in A_∞, we need to show that $A_\infty \models \exists \vec{y} \phi(\vec{a}, \vec{y})$. Using the directedness hypothesis, we see by an immediate induction that all the elements of the tuple $\vec{a}$ belongs to A_i for some $i \in I$. Hence, we have $A_i \models \exists \vec{y} \phi(\vec{a}, \vec{y})$. By definition, this means that there is some k-tuple $\vec{b}$ in A_i (which depends on $\vec{a}$) such that $A_i \models \phi(\vec{a}, \vec{b})$, which clearly implies that $A_\infty \models \phi(\vec{a}, \vec{b})$, because ϕ is quantifier-free. By definition, this means that

$$A_\infty \models \forall \vec{x} \exists \vec{y} \phi(\vec{x}, \vec{y}).$$

4. For any $n \in \mathbb{N}$, consider the poset $\langle n \rangle = \{0 < 1 < \cdots < n\}$ and the sequence of inclusion

$$\langle 0 \rangle \subseteq \langle 1 \rangle \subseteq \cdots$$

Each of the $\langle n \rangle$ has a top element, but the union is $\mathbb{N}$ with the usual order which clearly doesn't have a top element. Hence, by the previous question, the theory of posets with a top element in the language with one binary relation $\leq$ is not $\forall\exists$ axiomatizable.

Exercise 2. Let L be a language, M an L-structure and N a substructure of M. Show the following assertion:

If for every finite subset $K \subseteq N$ and every $a \in M$, there exists an automorphism α of M such that $\alpha(k) = k$ for every $k \in K$ and such that $\alpha(a) \in N$, then N is an elementary substructure of M.

Solution with the Tarski–Vaught test Consider an L-formula $\phi(y, \vec{x})$ with $n+1$ free variables and suppose that $M \models \exists y \phi(y, \vec{a})$, with $\vec{a} = (a_1, \ldots, a_n)$ an n-tuple of elements of N. By definition, this means that there exists $b \in M$ such that $M \models \phi(b, \vec{a})$. Now, let $K = \{a_1, a_2, \ldots, a_n\}$ and consider an automorphism α of M such that $\alpha(a_k) = a_k$ for $1 \leq k \leq n$ and such that $\alpha(b) \in N$. Since automorphisms (as any isomorphism) preserve the satisfiability of formulas, we have $M \models \phi(\alpha(b), a_1, \ldots, a_n)$. By the Tarski-Vaught test, this proves that N is an elementary substructure of M.

Solution without the Tarski–Vaught test This is similar to question 2 of the first exercise. We consider the following statement, denoted by H_k, which depends on k:

For all $n \in \mathbb{N}$, for every formula $\psi(\vec{y}, \vec{x})$ with $n + k$ free variables, for every n-tuple $\vec{a}$ of N,

$$\text{if } M \models Q_k y_k \ldots Q_1 y_1 \psi(y_1, \ldots, y_k, \vec{a}) \text{ then } N \models Q_k y_k \ldots Q_1 y_1 \psi(y_1, \ldots, y_k, \vec{a}).$$

If we can prove H_k for every $k \geq 0$, then this proves that N is an elementary substructure of M. We proceed by induction. If $k = 0$, the statement is true because N is a substructre of M by hypothesis. Suppose now that the statement H_k is true for a certain k and consider $n \geq 0$, a quantifier-free formula $\psi(y_{k+1}, y_k, \ldots, y_1, \vec{x})$ with $n + k + 1$ free variable, and an n-tuple $\vec{a}$ of N. Suppose that

$$M \models Q_{k+1} y_{k+1} \ldots Q_1 y_1 \psi(y_{k+1}, y_k, \ldots, y_1, \vec{a}),$$

for some sequence of quantifiers $Q_1, Q_2, \ldots, Q_{k+1}$. We need to prove that the same sentence holds in N. There are two cases to consider:

(a) If $Q_{k+1} = \forall$, let $b \in N$. We have

$$M \models Q_k y_k \ldots Q_1 y_1 \psi(b, y_k, \ldots, y_1, \vec{a}).$$

As the order of the argument in ψ has no importance and is purely notational, we can apply the induction hypothesis H_k with the $(n+1)$-tuple $(b, \vec{a})$ to deduce that $N \models Q_k y_k \ldots Q_1 y_1 \psi(b, y_k, \ldots, y_1, \vec{a})$. Since this is true for every $b \in N$, this means exactly that

$$N \models \forall y_{k+1} Q_k y_k \ldots Q_1 y_1 \psi(y_{k+1}, y_k, \ldots, y_1, \vec{a}).$$

(b) If $Q_{k+1} = \exists$, it means that there exists $b \in M$ such that

$$M \models Q_k y_k \ldots Q_1 y_1 \psi(b, y_k, \ldots, y_1, \vec{a}).$$

We can then pick an automorphism α of M such that $\alpha(b) \in N$ and such that $\alpha(a_i) = a_i$ for all $1 \leq i \leq n$. Since automorphisms preserve the truth of formulas, we have

$$M \models Q_k y_k \ldots Q_1 y_1 \psi(\alpha(b), y_k, \ldots, y_1, \vec{a}).$$

As in case (a), we can now apply the induction hypothesis H_k with $(n + 1)$-tuple $(\alpha(b), \vec{a})$, to get that $N \models Q_k y_k \ldots Q_1 y_1 \psi(\alpha(b), y_k, \ldots, y_1, \vec{a})$, which means, by definition, that

$$N \models \exists y_{k+1} Q_k y_k \ldots Q_1 y_1 \psi(y_{k+1}, y_k, \ldots, y_1, \vec{a}).$$

Exercise 3. Consider an infinite countable language L and let $\mathcal{F}_1$ be the set of L-formulas with at most one free variable. For a formula $\phi(x) \in \mathcal{F}_1$ and an L-structure M, the **value of ϕ in** M, denoted by $\mathrm{Val}(\phi, M)$, is the subset of M defined by formula ϕ. In other words, we have

$$\mathrm{Val}(\phi, M) = \{a \in M \mid M \models \phi(a)\}.$$

For an infinite cardinal λ, a **λ-structure** is an infinite L-structure M such that for every formula $\phi \in \mathcal{F}_1$, the value of ϕ in M is either a finite set or a set of cardinality λ. A model of a theory that is a λ-structure will be called a **λ-model**.

1. (1 pt) Show that if λ is an infinite cardinal, then every λ-structure has cardinality λ.
2. Let T be an L-theory and let $\phi(x)$ be a formula in $\mathcal{F}_1$. Suppose that for every integer n, T has a model in which the value of ϕ contains at least n elements.
   (a) (2pts) Show that for every infinite cardinal λ, T has at least one model in which the value of ϕ has cardinality at least λ.
   (*Hint:* Add to the language a set of constant symbols of cardinality λ.)

(b) (2 pts) Deduce that for every infinite cardinal λ, T has at least one model in which the value of ϕ has cardinality exactly λ.

3. (3 pts) Let T be an L-theory that has at least one infinite model. Prove that for every infinite cardinal λ, T has at least one λ-model.

   (*Hint:* Choose an infinite model M_0 of T and, with each formula $\psi \in \mathcal{F}_1$ whose value in M_0 is infinite, associate a set C_ψ of constant symbols of cardinality λ.)

4. (2 pts) Let S be a consistent L-theory that has only infinite models. Assume that for some infinite cardinal λ, all λ-models of S are isomorphic. Show that S is complete.

Solution

1. Consider the formula $x = x$. The valuation of this formula in M is simply the set M. Hence, if M is a λ-structure, then M is either a finite set or has cardinality λ. Since λ-structure are infinite by definition, M has cardinality λ.

2. (a) Add a set of new constant symbols C of cardinality λ to the language, and consider the theory in this new language
$$T_\phi = T \cup \{\phi(c) \,|\, c \in C\} \cup \{\neg(c = d) \,|\, c, d \in C\}.$$
If this theory has a model, then this provides a model of T such that the value of ϕ has cardinality at least that of C (which is λ). We can use the compactness theorem, and thus it suffices to show that for every finite subset $C_0 \subseteq C$, say of cardinality n, there exists a model of the theory
$$T' = T \cup \{\phi(c) \,|\, c \in C_0\} \cup \{\neg(c = d) \,|\, c, d \in C_0\}.$$
By hypothesis, we can choose a model of T in which the value of ϕ contains at least n elements and interpret the constants of C_0 as n distinct elements in the valuation of ϕ, hence providing a model of the theory T'.

   (b) We keep the notation from the answer to the previous question. Notice that since L is countable, the augmented language $L \cup C$ has cardinality λ. By the downward Löwenheim-Skolem theorem, there exists a model of T_ϕ which has cardinality $\|L \cup C\| = \lambda$. In particular, this provides a model of T of cardinality λ and such that the value of ϕ in M has cardinality at least λ. Hence the value of ϕ in M is exactly λ.

3. This is probably the hardest question of the whole exam. Let M_0 be an infinite model of T and let
$$A = \{\psi \in \mathcal{F}_1 \,|\, \text{the value of } \psi \text{ in } M_0 \text{ is infinite}\}.$$
For each $\psi \in A$, we choose an new set of constant symbols C_ψ of cardinality λ such that for $\psi \neq \phi \in A$, the sets C_ψ and C_ϕ are disjoint. We consider the new language L' obtained from L by adding all the sets C_ϕ for every $\psi \in A$. For $\psi \in A$, consider the theory
$$T_\psi = T \cup \{\psi(c) \,|\, c \in C_\psi\} \cup \{\neg(c = d) \,|\, c, d \in C_\psi\},$$
and let
$$T' = \mathrm{Th}(M_0) \cup \bigcup_{\psi \in A} T_\psi,$$
where $\mathrm{Th}(M_0)$ is the set of all L-sentences that hold in M_0. We first show that T' has a model. By the compactness theorem, it suffices to show that for every finite subset A_0 of A, and for every $\psi \in A_0$ and every finite subset D_ψ of D_ψ, the theory
$$T'' = \mathrm{Th}(M_0) \cup T \cup \bigcup_{\psi \in A_0} \{\psi(c) \,|\, c \in D_\psi\} \cup \{\neg(c = d) \,|\, c, d \in D_\psi\},$$
has a model. Clearly we can make M_0 a model of this theory by interpretating, for each $\psi \in A_0$, each constant symbol of D_ψ (which is finite) by a different element of $\mathrm{Val}(\psi, M)$, which by construction is infinite. Thus, T' has a model and, as in question 2, we can prove that it even has a model of cardinality λ. Let us denote by M such a model, which is in particular a model of T. All we have to prove now is that M is an λ-model of T. Let $\psi \in \mathcal{F}_1$. If $\psi \in A$, then the cardinality of $\mathrm{Val}(\psi, M)$ is infinite by construction. If $\psi \notin A$, then this means that $\mathrm{Val}(\psi, M_0)$ is finite. Let n be its cardinality and consider the L-sentence
$$\forall x_0 \forall x_1 \ldots \forall x_n \left(\bigwedge_{0 \le i \le n} \psi(x_i) \to \bigvee_{0 \le i \neq j \le n} (x_i = x_j) \right).$$
This sentence holds in M_0 (since the valuation of ψ in M_0 has cardinality n), and since N is a model of $\mathrm{Th}(M_0)$ by construction, this sentence also holds in N. Thus $\mathrm{Val}(\psi, N)$ is finite.

4. We argue by contradiction. Suppose that S is not complete, hence there exists an L-sentence ϕ such that both $S \cup \{\phi\}$ and $S \cup \{\neg\phi\}$ have a model. Because S only has infinite models, clearly $S \cup \{\phi\}$ and $S \cup \{\neg\phi\}$ can only have infinite models. By the previous question, for every infinite cardinal λ, both of these theories have an λ-model. By choosing a particular λ such that all λ-models of S are isomorphic, which exists by hypothesis, leads to a contradiction as this gives two isomorphic models of S such that ϕ holds in one of them and $\neg\phi$ in the other.

Exercise 4. Using proof trees, show the following statements.

1. (5 pts) $(\phi \wedge \psi) \to \chi \vdash \phi \to (\psi \to \chi)$
2. (5 pts) $\forall x\phi(x) \wedge \forall x\psi(x) \vdash \forall x\,(\phi(x) \wedge \psi(x))$

Exercise 5. Recall that for ordinals we use the notation $\alpha < \beta$ to say $\alpha \in \beta$, and, as usual, $\alpha \leq \beta$ means $\alpha < \beta$ or $\alpha = \beta$.

Let α and β be two ordinals. We say that β is cofinal with α if there exists a function $f : \beta \to \alpha$ whose image does not have a strict upper bound in α, that is, for every ξ in α, there exists δ in β such that $\xi \leq f(\delta)$. We define the *cofinality of an ordinal* α to be the least ordinal β such that β is cofinal with α. We denote it by $\mathrm{cof}(\alpha)$.

1. (2pts) Show that the definition of $\mathrm{cof}(\alpha)$ is well-defined, and show that we always have $\mathrm{cof}(\alpha) \leq \alpha$.
2. (2pts) What is the cofinality of a successor ordinal α? (Recall that this means that $\alpha = \beta + 1 (:= \beta \cup \{\beta\})$ for some ordinal β).

We say that an ordinal α is *regular* if $\mathrm{cof}(\alpha) = \alpha$.

3. (2pts) Show that ω is regular.
4. (2pts) Show that any regular ordinal is a cardinal.
5. (2pts) Let ω_1 be the first uncountable ordinal. Show that ω_1 is regular.

Solution

1. Consider the class C of ordinals β such that β is cofinal with α. By definition $\mathrm{cof}(\alpha)$ is the least element (for the order $\leq$) of this class, so we must check that this class is non-empty (recall that the well-orderedness of the class of all ordinals says that any *non empty* subclass of Ord has a least element).

   For that it suffices to notice that the identity function $\alpha \to \alpha$ is trivially cofinal, hence α is in the class C. This also shows that $\mathrm{cof}(\alpha) \leq \alpha$.

2. A successor ordinal α has a top element. Namely, if $\alpha = \beta \cup \{\beta\}$, then the top element is β. Indeed, any element $x \in \alpha$ is either $x = \beta$ or $x \in \beta$.

   Then, it suffices to notice that for an ordinal α with a top element, the function $1 \to \alpha$ which sends the unique element of 1 to the top element of α is cofinal. Hence $\mathrm{cof}(\alpha) \leq 1$. But clearly, the unique function $0 \to \alpha$ is not cofinal for a non-empty ordinal, so $\mathrm{cof}(\alpha) \neq 0$. All in all, this proves that $\mathrm{cof}(\alpha) = 1$ for any successor ordinal α.

3. Let n be a finite ordinal (which we can think of as a natural integer) and $f : n \to \omega$. Any finite set of natural integers has a maximum (which is itself an integer), hence the image of f has a strict upper bound, for example the previous maximum $+1$. This proves that no function $f : n \to \omega$ can be cofinal (i.e. having the property saying that n is cofinal with ω). Combined with the fact that $\mathrm{cof}(\omega) \leq \omega$ (from question 1), this proves that $\mathrm{cof}(\omega) = \omega$.

4. Recall that an ordinal α is a cardinal if it is the smallest ordinal of its cardinality. In other words, α is a cardinal if for every ordinal β such that $\beta < \alpha$, then there is no bijection between α and β. Equivalently, this means that if β and α are in bijection, then $\beta \geq \alpha$.

   Notice also that if there is a surjection (and in particular a bijection) $\beta \to \alpha$, then the property given at the beginning of the exercise is trivially true, hence β is cofinal with α.

   Now, let α be a regular ordinal, and let β be another ordinal and suppose that there exists a bijection $g : \beta \to \alpha$. In particular, β is cofinal with α as previously said, hence $\alpha = \mathrm{cof}(\alpha) \leq \beta$, where the first equality comes from the fact that α is regular.

5. This is similar to question 3. Let $f : \beta \to \omega_1$ be cofinal with α a countable ordinal. Since a countable union of countable sets is countable, this proves that the supremum of the image of f is countable. Thus, it necessarily has a strict upper bound in ω_1. This shows that no countable ordinal is cofinal with ω_1; combined with the fact that $\mathrm{cof}(\omega_1) \leq \omega_1$, this proves that $\mathrm{cof}(\omega_1) = \omega_1$.