Midterm Exam: Grondslagen van de Wiskunde

December 16, 2025, 09:00-12:00

This exam consists of 5 exercises; see also the back side.
Every exercise is worth 10 points; your grade is the total divided by 5. If an exercise contains several questions, the number of points attributed to each question is indicated.
As the exam is **closed book**, some basic definitions about posets, which you may use freely, are recalled below.
Advice: start first with the exercises that you can do, and then think about the rest. Succes!

A few definitions about posets: Recall that a poset is a pair $(X, \leq)$ of a set X and a binary relation $\leq$ such that:

- (reflexivity) $x \leq x$ for any $x \in X$
- (anti-symmetry) $x \leq y$ and $y \leq x$ imply $x = y$ for any $x, y \in X$,
- (transitivity) $x \leq y$ and $y \leq z$ imply $x \leq z$, for any $x, y, z \in X$.

We say furthermore that the order is *linear* (or *total*) if for any $x, y \in X$, we have $x \leq y$ or $y \leq x$.
We will often only refer to a poset as X instead of $(X, \leq)$.
A poset is *well-ordered* if every non-empty subset has a least element.
A function $f\colon L \to M$ between well-ordered sets is an *embedding* if it is strictly increasing (see Exercise 1) and for any $x \in L$ and any $y \in M$ such that $y \leq f(x)$, there exists $x' \leq x$ with $f(x') = y$.

Exercise 1.

1. (2 points) Let P and Q be posets and $f\colon P \to Q$ a function. Prove that if P is linearly ordered, then the following assertions are equivalent:
   (a) f is order-preserving (if $x \leq y$ then $f(x) \leq f(y)$) and injective,
   (b) f is strictly increasing: if $x < y$ then $f(x) < f(y)$,
   (c) f satisfies the condition: $x < y$ if and only if $f(x) < f(y)$.
2. (4 points) Let L and M two well-ordered sets, $f\colon L \to M$ an embedding and $g\colon L \to M$ a strictly increasing function. Prove that for every $x \in L$, we have
$$f(x) \leq g(x).$$
3. (4 points) Let L and M be two well-ordered sets, prove that there exists an embedding $L \to M$ if and only if there exist a strictly increasing function $L \to M$.

Solution

1. $(a) \Rightarrow (b)$: Suppose that f is order-preserving and injective, and let $x < y$. Since f is order preserving we have $f(x) \leq f(y)$, but since f is injective we have $f(x) \neq f(y)$, hence $f(x) < f(y)$.

   $(b) \Rightarrow (a)$ Suppose that f is strictly increasing. Clearly f is order-preserving, hence we have to prove that f is injective. Let $x, y \in P$ such that $f(x) = f(y)$. Since P is linearly ordered, we either have $x > y$ or $x < y$ or $x = y$. Using that f is strictly increasing, we see that the only possibility is $x = y$.

   $(b) \Rightarrow (c)$: Suppose that f is strictly increasing and let $x, y \in P$ such that $f(x) < f(y)$. Suppose by contradiction that $x \not< y$. Because P is linearly ordered, this means that necessarily $x \geq y$. Using that f is increasing, this implies that $f(x) \geq f(y)$ which is a contradiction. Hence, we have $x < y$.

   $(c) \Rightarrow (b)$ trivial.
2. Let $A \subseteq L$ be the set of those x such that $f(x) \leq g(x)$. Since f is an embedding, we have $f(0) = 0$, hence $0 \in A$. Now let $x \neq 0 \in L$ such that $L_{<x} \subseteq A$, there are two cases:

- if x is a successor, $x = y + 1$. Because g is strictly increasing, we have $g(y) < g(x)$, hence $g(y) + 1 \leq g(x)$. Because f is an embedding, we have $f(x) = f(y) + 1$. By hypothesis, $y \in A$, hence $f(y) \leq g(y)$. All in all, we have
$$f(x) = f(y) + 1 \leq g(y) + 1 \leq g(x).$$
- if x is limit, then $x = \sup\{y < x\}$. Because g is strictly increasing, we have $g(y) < g(x)$ for all $y < x$. In particular, $\sup\{g(y) \mid y < x\} \leq g(x)$. Now because f is an embedding, we have $f(x) = \sup\{f(y) \mid y < x\}$, hence
$$f(x) = \sup\{f(y) \mid y < x\} \leq \sup\{g(y) \mid y < x\} \leq g(x),$$
where the first inequality comes from the fact that $L_{<x} \subseteq A$, hence for any $y < x$, we have $f(y) \leq g(y)$.

3. The "only if" part is trivial, since, by definition, an embedding is strictly increasing. Let us prove the "if" part. Let $g\colon L \to M$ be a strictly increasing function. We are going to define an embedding $f\colon L \to M$ be transfinite recursion. We set $f(0) = 0$. Now, let $x \in L$, with $x \neq 0$, such that we have define an embedding $f|_{L_{<x}}\colon L_{<x} \to M$, we need to give a rule to define $f(x)$. There are two cases:
   - x is a successor, $x = y + 1$, then we have to define $f(x) = f(y) + 1$, but to do that we need to be sure that $f(x)$ is not a greatest element (and thus that the successor of $f(x)$ does exist). Applying question 2 of the exercise with $L_{<x}$ in place of L, we have that $f(y) \leq g(y)$, and because g is strictly increasing, we have $g(y) < g(x)$, which proves that $f(y)$ is not a greatest element.
   - x is limit, then $x = \sup\{y < x\}$, then we have to define $f(x) = \sup\{f(y) \mid y < x\}$, but to do that we need to be sure that this supremum exist, hence we need to show that the set $\{f(y) \mid y < x\}$ has an upper bound. Applying question 2 of the exercise with $L_{<x}$ in place of L, we have that $f(y) \leq g(y)$, for every $y < x$. Since, g is strictly increasing, $g(x)$ is an upper bound of the set $\{g(y) \mid y < x\}$ and thus of the set $\{f(y) \mid y < x\}$.

   It is immediate to check that the function f such defined is an embedding.

Exercise 2. A *tree* is a poset $(T, \leq)$ such that:

(i) it has a least element, called the *root*,

(ii) for every $x \in T$, the set $\{y \in T \mid y \leq x\}$ is well-ordered.

Given an element $x \in T$, an *immediate successor* of x is an element y such that $x < y$ and there is no z such that $x < z < y$. We say that a tree is *infinite* if the underlying set is infinite, and we say that it is *locally finite* if for every element $x \in T$, the set of immediate successors of x is finite.

1. (3 points) Let $(T, \leq)$ be an infinite tree which is locally finite. Prove that if x is an element such that the set $\{y \in T \mid y \geq x\}$ is infinite, then there is an immediate successor of x having the same property.
2. (7 points) Prove the famous Kőnig's lemma: For every infinite and locally finite tree, there exists an infinite strictly increasing sequence
$$x_0 < x_1 < x_2 < \ldots.$$
You have to make explicit where you use the axiom of choice in your answer.

Solution

1. Let us first prove the following fact: if $x < y$, then $x' \leq y$ for some x', immediate successor of x. For that, it suffices to notice that the set $\{z \leq y \mid z \in T\}$ is well-ordered by definition, hence, because x is in that set and is not a greatest element, it has a successor $x + 1$ in that set. It is then trivial to check that this element is an immediate successor of x in T.

   From this fact it follows that for any $x \in T$, we have
$$\{y \geq x \mid y \in T\} = \{x\} \cup \bigcup_{x' \in \mathrm{Succ}(x)} \{y \geq x' \mid y \in T\}.$$
   Where $\mathrm{Succ}(x)$ is the set of all the immediate successors of x. By hypothesis, there are only a finite number of these, and since a finite union of finite sets is finite, if $\{y \geq x \mid y \in T\}$ is infinite, then necessarily one of the set $\{y \geq x' \mid y \in T\}$ has to be infinite.

2. Notice that if such a sequence exist, then for every x_n, then the set $\{y \geq x_n \mid y \in T\}$ is infinite. So, we are going to construct by recursion such a sequence with this property. We define x_0 to be the root, and if the sequence has be defined up to x_n for some $n \in \mathbb{N}$, (with the property that $\{y \geq x_n \mid y \in T\}$ is infinite), then we define x_{n+1} to be any of its immediate successor whose sets of greater elements is infinite. The existence of such an immediate successor follows from the previous question.

   The use of the axiom of (countable) choice here follows from the fact that at each step we have to make a choice, because there can be more than one immediate successor of x_n having the desired property. In particular, we need to make a (countable) infinite number of choices, hence the use of the axiom of choice.

Exercise 3. Let X be an infinite set and let $A \subseteq X^X$ an *infinite* set of functions $X \to X$. Let $\overline{A}$ be the smallest subset of X^X such that: $A \subseteq \overline{A}$ and $\overline{A}$ is closed under composition, i.e. if $f, g \in \overline{A}$, then $f \circ g \in \overline{A}$.

Show that $|\overline{A}| = |A|$.

Solution Obviously, we have $|A| \leq |\overline{A}|$. For the other inequality, notice that any element of $\overline{A}$ can be represented as a finite sequence $(f_1, f_2, \ldots, f_n)$ of elements of A. Hence, we have

$$|\overline{A}| \leq \sum_{n>0} |A|^n.$$

Because, A is infinite we have $|A|^n = |A|$ for any $n > 0$, and thus

$$\sum_{n>0} |A|^n = \sum_{n>0} |A| = \omega \times |A| \leq |A| \times |A| = |A|,$$

where the last inequality and last equality both comes from the fact that A is infinite. Hence, we have $|\overline{A}| \leq |A|$. By Cantor-Schröder-Bernstein, we conclude that $|A| = |\overline{A}|$.

Exercise 4. Consider the language $L = \{f\}$, with f is a unary function symbol, and the L-structure R, whose underlying set is $\mathbb{R}$, and $f^R(x) = x^2$.

1. (5 points) Using L-formulas, define in R the numbers $-1, 0$ and 1. That is, give L-formulas with one free variable such that the set of elements of R for which the formula holds is $\{-1\}$ (resp. $\{0\}$, resp. $\{1\}$).
2. (5 points) Find an L-formula $\phi(x)$ such that the subset of $x \in R$ for which $\phi(x)$ holds is the subset $\{x \in \mathbb{R} \mid x > 0\}$.

Solution

1. We can use the formulas

$$\begin{aligned} \phi_{-1}(x) \equiv &\quad (f(f(x)) = f(x)) \wedge \neg\,(f(x) = x)\,, \\ \phi_0(x) \equiv &\quad \forall y\,((f(y) = x) \leftrightarrow (y = x))\,, \\ \phi_1(x) \equiv &\quad ((\phi(x) = x) \wedge \neg\phi_0(x))\,. \end{aligned}$$

2. We can use the formula

$$\phi_{>0}(x) \equiv \exists y\,((f(x) = y) \wedge \neg\phi_0(y))\,.$$

Exercise 5. Let $(X, \leq)$ be a poset. The goal of the exercise is to prove that there always exists a linear order $\preceq$ on X that extends $\leq$ (i.e. $x \leq y$ implies $x \preceq y$).

1. (4 points) Prove the statement in the case that X is finite.
2. (6 points) Prove the general case using the compactness theorem.

Solution

1. We proceed by induction on the cardinality n of X. If $n = 0$, then the assertion is trivial. Suppose proven the statement for $n \geq 0$ and let X be of cardinality $n + 1$. Pick any $x \in X$. By hypothesis there exists a linear order $\preceq$ on $X - \{x\}$ which extends $\leq$. Without loss of generality, we can denote the elements of $X - \{x\}$ by $x_1, x_2, \ldots, x_n$ according to the linear order, i.e. with

$$x_1 \prec x_2 \prec \cdots \prec x_n.$$

Now consider the set $A = \{y \in X - \{x\} \mid y \not< x\}$. We have $A \subseteq X - \{x\}$. If A is empty, this means that for every element $y \neq x$ of X, either $y < x$, or y and x are not comparable (i.e. neither $x > y$, nor $x < y$). In particular, we extend $\preceq$ from $X - \{x\}$ to X by asserting that x is the greatest element of X for $\preceq$. It is straightforward to see that $\preceq$ on X extends $\leq$.

If A is not empty, then it has a lowest element x_k. If $k = 0$, then we extend $\preceq$ from $X - \{x\}$ to X by asserting that x is the least element of X for $\preceq$, otherwise, we just "insert" x between x_{k-1} and x_k:

$$x_{k-1} \prec x \prec x_k.$$

It is then straightforward to check that $\preceq$ extends $\leq$ on X.

2. Consider the language L_X which has one binary relation symbol $\preceq$ and a constant c_x for every $x \in X$, and consider the theory T_X which contains the theory of linearly ordered sets T_{lin} as well as the set of axioms

$$\{c_x \prec c_y \mid \text{ for all } x < y \in X\}.$$

If we can prove that T_X has a model M, then the function $X \to M, x \mapsto c_x$ is strictly increasing and taking the order on X induced by that of M proves exactly what we want. By the compactness theorem, to prove that such a model exists, it suffices to prove that every finite subtheory $T' \subseteq T_X$ has a model. But for any such theory we have

$$T' \subset T_{\text{lin}} \cup \{c_x \prec c_y \mid \text{ for a finite set of } x < y \text{ in } X\}.$$

The answer to question 1 proves in particular that the theory on the right has a model, q.e.d.