Simplicial cheat-sheet

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1 Simplicial and co-simplicial objects

1.1. For any non-negative integer n, we denote by [n] the ordered set

$$\{0 < 1 < .. < n\}$$

(for n = 0, this is just a point). The category Δ is the category whose objects are all the [n] for $n \ge 0$ and whose morphisms are non-decreasing functions between these ordered sets. For $n \ge 0$ and $0 \le i \le n$, we denote by $\delta^i : [n-1] \to [n]$ the only non-decreasing injective function such that i is not in its image. Explicitly, we have

$$\delta^{i}(k) = \begin{cases} k & \text{if } k < i, \\ k+1 & \text{if } k \ge i. \end{cases}$$

The δ^i 's are referred to as the co-face maps. Similarly, for $n \ge 0$ and $0 \le i \le n$, we denote by $\sigma^i \colon [n+1] \to [n]$ the only non-decreasing surjective function such that hits *i* twice. Explicitly, we have

$$\sigma^{i}(k) = \begin{cases} k & \text{if } k \leq i, \\ k-1 & \text{if } k > i. \end{cases}$$

The σ^i 's are referred to as the co-degeneracy maps. As is easily checked, we have the following so-called *co-simplicial identities*

$$\begin{split} \delta^{j} \delta^{i} &= \delta^{i} \delta^{j-1} \text{ if } i < j \\ \sigma^{j} \delta^{i} &= \delta^{i} \sigma^{j-1} \text{ if } i < j \\ id &= \sigma^{j} \delta^{j} = \sigma^{j} \delta^{j-1} \\ \delta^{j} \sigma^{i} &= \delta^{i-1} \sigma^{j} \text{ if } i > j+1 \\ \sigma^{j} \sigma^{i} &= \sigma^{i} \sigma^{j+1} \text{ if } i \leq j. \end{split}$$

As one can prove, Δ is in fact the category freely generated by the morphisms δ^i and σ^i and subject to the co-simplicial identities.

Definition 1.2. A simplicial object X in a category C is a functor

$$X: \Delta^{\mathrm{op}} \to \mathcal{C}.$$

A morphism $X \to X'$ of simplicial objects is simply a natural transformation. We denote by $s\mathcal{C}$ the category of simplicial objects in \mathcal{C} . Dually, a co-simplicial object Y in \mathcal{C} is a functor

$$Y\colon \Delta \to \mathcal{C},$$

and morphisms of co-simplicial objects are natural transformations. We denote by $\cos C$ the category of simplicial objects.¹

Remark 1.3. In practise, a co-simplicial object $Y \colon \Delta \to \mathcal{C}$ consists of the data of

- a sequence $(Y^n)_{n>0}$ of objects of \mathcal{C}
- morphisms $\partial^i \colon Y^n \to Y^{n+1}$ for all $n \ge 0$ and $0 \le i \le n+1$,
- morphisms $s^i \colon Y^{n+1} \to Y^n$ for all $n \ge 0$ and $0 \le i \le n$,

satisfying the co-simplicial identities. Dually, a simplicial object $X \colon \Delta^{\mathrm{op}} \to \mathcal{C}$ consists of the data of

- a sequence $(X_n)_{n>0}$ of objects of \mathcal{C} ,
- morphisms $\partial_i \colon X_n \to X_{n+1}$ for all $n \ge 0$ and $0 \le i \le n+1$,
- morphisms $s_i \colon X_{n+1} \to X_n$ for all $n \ge 0$ and $0 \le i \le n$,

satisfying the simplicial identities (i.e. the dual of the co-simplicial identities).

Remark 1.4. Note that we have

$$\cos \mathcal{C} = (s(\mathcal{C}^{\mathrm{op}}))^{\mathrm{op}}.$$

Example 1.5. The case of C = Set is fundamental, the objects of sSet are referred to as *simplicial sets*. Note that the Yoneda embedding defines a co-simplicial object in sSet

$$y_{\Delta} \colon \Delta \to \mathsf{sSet}.$$

Usually, we use the notation

$$\Delta[n] := y_{\Delta}[n].$$

Example 1.6. Since we can see any ordered set as a category and any nondecreasing function as a functor, we have a canonical functor

$$i \colon \Delta \to \mathsf{Cat},$$

where Cat is the category of small categories. This defines a co-simplicial object in Cat. Observe that this functor is fully faithful, and is often identified with a full subcategory inclusion.

Example 1.7. For any $n \ge 0$, let Δ^n be the standard topological *n*-simplex, that is Δ^n is the convex hull in \mathbb{R}^n of the standard basis (e_0, e_1, \dots, e_n) of \mathbb{R}^n . One defines co-face maps as

$$\begin{split} \delta^i &: \Delta^{n-1} \to \Delta^n \\ & \sum_{j=0}^{n-1} t_j e_j \mapsto \left(\sum_{j=0}^{i-1} t_j e_j + \sum_{j=i+1}^{n+1} t_{j-1} e_j \right), \end{split}$$

¹The notation $\cos C$ is not standard.

and co-degeneracy maps as

$$\sigma^{i} \colon \Delta^{n+1} \to \Delta^{n}$$

$$\sum_{j=0}^{n+1} t_{j} e_{j} \mapsto \left(\sum_{j=0}^{i-1} t_{j} e_{j} + (t_{i} + t_{i+1}) e_{i} + \sum_{j=i+1}^{n} t_{j+1} e_{j} \right).$$

One checks (exercise) that the co-simplicial identities are satisfied and thus defines a co-simplicial object in the category **Top** of topological spaces

$$\Delta \to \mathsf{Top}$$
$$[n] \mapsto \Delta^n.$$

As it turns out, co-simplicial objects allow to define functors into the category of simplicial sets.

1.8. Let $i: \Delta \to C$ a co-simplicial object in a category C. We denote by i^* the functor from C to the category of simplicial sets sSet (i.e. simplicial objects in Set) defined as

$$i^* \colon \mathcal{C} \to \mathsf{sSet}$$

 $X \mapsto ([n] \mapsto \operatorname{Hom}_{\mathcal{C}}(i[n], X))$

If C is co-complete (i.e. has all colimits), then i^* has a left adjoint $i_!$ defined by

$$i_! : \mathsf{sSet} \to \mathcal{C}$$

 $X \mapsto \operatornamewithlimits{colim}_{([n], \Delta[n] \to X)} i[n]$

Example 1.9. If we consider $i: \Delta \to \mathsf{Cat}$, then the functor i^* is often denoted as N. For a small category C, N(C) is referred to as the *nerve of* C.

Example 1.10. Consider the co-simplicial object $i: \Delta \to \mathsf{Top}, [n] \mapsto \Delta^n$. Then the functor i^* is often denoted as Sing. For a topological space X, SingX is referred to as the singular complex of X.

2 Simplicial objects and chain complexes

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2.1. Let \mathcal{A} be an abelian category. Given a simplicial objet $X : \Delta^{\mathrm{op}} \to \mathcal{A}$, we define a chain complex $C_{\bullet}X$ as

$$C_k X := X_k$$

and the boundary map is given by

$$: C_k X \to C_{k-1} X$$
$$x \mapsto \sum_{i=0}^n (-1)^i \partial_i(x).$$

One indeed checks using the simplicial identities that we have $\partial \circ \partial = 0$. This construction canonically defines a functor

$$C \colon s\mathcal{A} \to \mathsf{Ch}_{\geq 0}(\mathcal{A})$$
$$X \mapsto C_{\bullet}X.$$

2.2. In the definition of the functor $C_{\bullet}: s \mathcal{A} \to \mathsf{Ch}_{\geq 0}(s \mathcal{A})$, we haven't make use of the degeneracy maps. We can actually use them to refine the functor C_{\bullet} as follows. For $X: \Delta^{\mathrm{op}} \to \mathcal{A}$, we say that a *n*-simplex $x \in X_n$, with $n \geq 1$, is *degenerate* if it is in the image of one of the $s_i: X_{n-1} \to X_n$. We denote by $D_n X$ the subobject of X_n generated by degenerated simplices, and we define $N_n X$ as

$$N_n X := C_n X / D_n X.$$

One can check that the differential $\partial \colon C_n X \to C_{n-1} X$ induces a morphism

$$\partial: N_n X \to N_{n-1} X.$$

The chain complex $N_{\bullet}X$ is called the *normalized chain complex* associated to X. This construction extends naturally to a functor

$$N_{\bullet} : s\mathcal{A} \to \mathsf{Ch}_{>0}(s\mathcal{A}).$$

By construction, we have a natural transformation

$$p\colon C_{\bullet} \Rightarrow N_{\bullet}.$$

Proposition 1. Let \mathcal{A} be an abelian category. For every object X of $s \mathcal{A}$, the morphism of chain complexes

$$p_X \colon C_{\bullet}X \to N_{\bullet}X$$

is a quasi-isomorphism.

Proof. (Sketch) By construction, we have a short exact sequence of chain complexes

$$0 \to D_{\bullet}X \to C_{\bullet}X \to N_{\bullet}X \to 0.$$

Using simplicial combinatorics, one can show that the homology of $D_{\bullet}X$ is trivial. The result follows them from the long exact sequence on homology. \Box

The main interest of the normalized chain complex functor comes from the following theorem, known as the Dold–Kan correspondence.

Theorem 2.3. (Dold-Kan correspondence) Let s A be an abelian category. The functor

$$N_{\bullet} : s\mathcal{A} \to \mathsf{Ch}_{\geq 0}\mathcal{A}$$

is an equivalence of categories.

Remark 2.4. Since the opposite category of an abelian category is an abelian category, all the previous results can be dualized. In particular, if $s \mathcal{A}$ is an abelian category, we have an equivalence of categories between co-simplicial objects in \mathcal{A} and non-negatively graded cochain complexes in \mathcal{A}

$$\cos \mathcal{A} \simeq \operatorname{coCh}^{\geq 0}(\mathcal{A}).$$

2.5. In light of the Dold–Kan correspondence, we see that looking at simplicial (resp. co-simplicial) objects in an abelian category is the same thing as looking at non-negatively graded chain complexes (resp. cochain complexes) in \mathcal{A} . The point is that the notion of simplicial and co-simplicial objects in a category makes sense even if the category is not pre-additive. In other words, looking at (co)-simplicial objects in a (nice) category is a way to do "non-abelian homological algebra". Of course, we are particularly interested in the case that the ambient category is a topos \mathcal{E} , hence giving rise to the notion of (co)simplicial sheaves.